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SPECIAL SECTION: GEOMETRY IN ALGEBRA AND ALGEBRA IN GEOMETRY

Liouvillian solutions for second order linear differential equations with polynomial coefficients

Primitivo B. Acosta-Humánez[1,2] · **David Blázquez-Sanz**[3] · **Henock Venegas-Gómez**[3]

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Abstract
In this paper we present an algebraic study concerning the general second order linear differential equation with polynomial coefficients. By means of Kovacic's algorithm and asymptotic iteration method we find a degree independent algebraic description of the spectral set: the subset, in the parameter space, of Liouville integrable differential equations. For each fixed degree, we prove that the spectral set is a countable union of non accumulating algebraic varieties. This algebraic description of the spectral set allow us to bound the number of eigenvalues for algebraically quasi-solvable potentials in the Schrödinger equation.

Keywords Anharmonic oscillators · Asymptotic iteration method · Kovacic algorithm · Liouvillian solutions · Parameter space · Quasi-solvable model · Schrödinger equation · Spectral varieties

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1 Introduction

Let us consider the family of second order linear differential equations,

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✉ David Blázquez-Sanz
dblazquezs@unal.edu.co

1 Instituto Superior de Formación Docente Salomé Ureña, Santiago, Dominican Republic

2 Fac. Ciencias Básicas y Biomédicas, Universidad Simón Bolívar, Barranquilla, Colombia

3 Escuela de Matemáticas, Facultad de Ciencias, Universidad Nacional de Colombia - Sede Medellín, Medellín, Colombia

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$$u'' + P(x)u' + Q(x)u = 0, \tag{1}$$

with polynomial coefficients of bounded degree. This family is parameterized by the coefficients of P and Q and therefore endowed of an structure of affine algebraic variety. We are interested in characterizing the moduli of Liouville integrable diferentential equations in (1) and describing how the Liouvillian solutions of those integrable equations depend on the coefficients. From a result of Singer [17], we expect that this moduli to be enumerable union of constructible set corresponding to possible choices of local exponents at infinity of Liouvillian solutions.

With this purpose we explore the application of Kovacic's algorithm (see [10, 12]) to the family (1). Some steps of the algorithm, dealing with polynomial solutions of auxiliar equations, are very sensitive to changes of the parameters. However, the Asymptotic Iteration Method (see [7]) allows us to describe the algebraic conditions on the parameters giving rise to the existence of Liouvillian solutions.

The structure of the paper is as follows. Section 2 is devoted to the definitions of parameter space $\mathbb{P}_{2n}$, spectral set $\mathbb{L}_{2n}$, spectral varieties $\mathbb{L}_{2n,d}$ and the statement of our first main result, Theorem 2.3. Section 3 is devoted to the definition of polynomial-hyperexponential solutions, the reduction of the parameter space through D'Alembert transformation, and Kovacic's algorithm. The analysic of equation $y'' = (x^{2n} + \mu x^{n-1})y$ allows us to prove the non-emptyness of the spectral varieties $\mathbb{L}_{2n,kn}$ and $\mathbb{L}_{2n,kn+1}$ (Corolary 3.6). Section 4 contains the results of this paper related to Asymptotic Iteration Method. We find a sequence of differential polynomials $\Delta_d(a, b)$ in two variables that codify the equations of the spectral varieties $\mathbb{L}_{2n,d}$ inpendently of n (Theorem 4.3). The proof of Theorem 2.3 is included at the end of the section. Section 5 is devoted to the Liovullian solutions of Schrödinger equations with polynomial potentials. We proof that the number of the values of the energy parameter allowing a Liouvillian eigenfunction is bounded by *the arithmetic condition* which is a simple function of the coefficients of the potential (Theorem 5.2).

2 Parameter space

Let us consider Eq. (1) with polynomial coefficients $P(x) = \sum_{j=1}^{n} p_j x^j \in \mathbb{C}[x]_{\leq n}$ and $Q(x) = \sum_{j=1}^{2n} q_j x^j \in \mathbb{C}[x]_{\leq 2n}$. We also take into account a non-degeneracy condition $p_n^2 - q_{2n} \neq 0$, which implies that the equation can be reduced to trace free form with a polynomial coefficient of degree $2n$. Thus, the *parameter space* corresponding to the family of Eq. (1) is,

$$\mathbb{P}_{2n} = (\mathbb{C}[x]_{\leq n} \times \in \mathbb{C}[x]_{\leq 2n}) - \{p_n^2 - q_{2n} = 0\},$$

that we consider an an affine algebraic variety of dimension $3n + 2$ with affine coordinates $p_0, \dots, p_n, q_0, \dots, q_{2n}$. Our purpose is to describe algebraically the *spectral set* $\mathbb{L}_{2n} \subseteq \mathbb{P}_{2n}$. That is, the set of equations in the family (1) admitting a Liouvillian solution. An important class of Liouvillian functions, specifically relevant for the integrability of Eq. (1) is the following.

Definition 2.1 A polynomial-hyperexponential function of polynomial degree d and exponential degree k is a function of the form

$$u(x) = P_d(x)e^{\int A_k(x)dx}, \tag{2}$$

with $P_d(x)$ and $A_k(x)$ polynomials of degree d and k respectively.

Definition 2.2 The spectral subvariety $\mathbb{L}_{2n,d}$ is the subset of $\mathbb{L}_{2n}$ corresponding to equations in the family (1) having a polynomial-hyperexponentional solution of polynomial degree d.

Theorem 2.3 *Let $\mathbb{L}_{2n} \subset \mathbb{P}_{2n}$ be the set of equations in family (1) having a Liouvillian solution, and $\mathbb{L}_{2n,d}$ be the set of equations in family (1) having a polynomial-hyperexponential solution of polynomial degree d . The following statements hold:*

(a) *For any fixed $n \in \mathbb{N}$ there is an infinite set of values of d such that $\mathbb{L}_{2n,d}$ is not empty.*
(b) *If not empty, $\mathbb{L}_{2n,d}$ is an algebraic variety of codimension $\leq n$ in $\mathbb{P}_{2n}$.*
(c) *For any $d \neq k$ the algebraic varieties $\mathbb{L}_{2n,d}$ and $\mathbb{L}_{2n,k}$ are disjoint in $\mathbb{P}_{2n}$.*
(d) *Any compact subset of $\mathbb{P}_{2n}$ intersects only a finite number of algebraic varieties of the family $\{\mathbb{L}_{2n,d}\}_{d\in\mathbb{N}}$.*

Furthermore,

$$\mathbb{L}_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}_{2n,d}.$$

Therefore we conclude that $\mathbb{L}_{2n}$ is a singular analytic submanifold of $\mathbb{P}_{2n}$ consisting in the enumerable union of pairwise disjoint algebraic varieties of codimension $\leq n$ in $\mathbb{P}_{2n}$.

In what follows we will deal with the proof of Theorem 2.3 and the calculation of the equations of the spectral subvarieties $\mathbb{L}_{2n,d}$ in suitable coordinates.

3 Liouvillian solutions

3.1 Reduction of the parameter space

As it is well known, Eq. (1) can be reduced to trace free form

$$y'' = R(x)y \tag{3}$$

by means of D'Alembert transform $u = \exp\left(-\frac{1}{2}\int P(x)dx\right)y$, where $R(x) = \frac{P(x)^2}{4} + \frac{P'(x)}{2} - Q(x)$. Note that the degree of $R(x)$ is not greater than $\max\{\deg(Q(x)), 2\deg(P(x))\}$. Note that the family of equations of the form (3) with

$R(x)$ of fixed degree $2n$ are parameterized by the space $\mathbb{K}_{2n} = \mathbb{C}[x]_{2n}$ of polynomials of degree $2n$ that we see as an affine algebraic variety of dimension $2n+1$, parameterized by the coefficients of $R(x)$ and thus isomorphic to $\mathbb{C}^* \times \mathbb{C}^{2n}$. Note that the family (3) is included in (1), where $R(x) \in \mathbb{K}_{2n}$ corresponds to $(0, -R(x)) \in \mathbb{P}_{2n}$. The D'Alembert transformation is a polynomial map in the coefficients of $P(x)$ and $Q(x)$ and it can be seen as a retract,

$$\mathrm{dal}_{2n} : \mathbb{P}_{2n} \to \mathbb{K}_{2n} \subset \mathbb{P}_{2n}, \quad (P(x), Q(x)) \mapsto R(x) = \frac{P(x)^2}{4} + \frac{P'(x)}{2} - Q(x) \mapsto (0, -R(x))$$

of the natural inclusion $\mathbb{K}_{2n} \subset \mathbb{P}_{2n}$. Taking into account that the ratio between u and y is the exponential of a polynomial, we obtain that $(P(x), Q(x)) \in \mathbb{L}_{2n,d}$ if and only if $R(x) \in \mathbb{L}_{2n,d} \cap \mathbb{K}_{2n}$. Therefore, the analysis of polynomial-hyperexponential solutions of a given polynomial degree can be restricted to the trace free family $\mathbb{K}_{2n}$.

Let us write $R(x) = \sum_{j=0}^{2n} r_j x^j$. Equation (3) can be reduced to the case of monic polynomial coefficient by the change of variables $x \mapsto \sqrt[2n+2]{\frac{1}{r_{2n}}}x$ which lead us to the equation

$$y'' = \left(x^{2n} + \sum_{j=0}^{2n-1} \frac{r_j}{\sqrt[2n+2]{\frac{1}{r_{2n}}}} x^j \right) y, a_k \in \mathbb{C}^*, a_i \in \mathbb{C}. \tag{4}$$

For the next step, let us consider $\mathbb{M}_{2n} \subset \mathbb{K}_{2n}$ the family of Eq. (3) with monic polynomial coefficient. It is an an algebraic variety isomorphic to $\mathbb{C}^{2n}$. Since the $(2n+2)$-th root of r_{2n} is an algebraic multivalued function of r_{2n}, any equation in $\mathbb{K}_{2n}$ has $2n+2$ different equivalent reductions in $\mathbb{M}_{2n}$. This can be seen as an algebraic correspondence $\mathcal{C}_{2n} \subset \mathbb{K}_{2n} \times \mathbb{M}_{2n}$. This algebraic correspondence is a $(2n+2)$-fold covering space of $\mathbb{K}_{2n}$ by the first projection, π_1 and the $(2n+2)$ monic reductions of the equation of coefficient $R(x)$ are given by $\pi_2(\pi_1^{-1}(\{R(x)\}))$. Note that $R(x)$ is in $\mathbb{L}_{2n,d}$ if and only if so are any of its monic reductions. Therefore, if suffices to focus our analysis to equations in the family $\mathbb{M}_{2n}$.

3.2 Kovacic's algorithm and adapted coordinates in $\mathbb{M}_{2n}$

From now on let $\mathbb{L}'_{2n} = \mathbb{L}_{2n} \cap \mathbb{M}_{2n}$ be the *reduced spectral set* consisting of equations in $\mathbb{M}_{2n}$ having a Liouvillian solution, and let $\mathbb{L}'_{2n,d} = \mathbb{L}_{2n,d} \cap \mathbb{M}_{2n}$ be the *reduced spectral variety* consisting of equations in $\mathbb{M}_{2n}$ having a polynomial-hyperexponential solution of polynomial degree d.

Note that, since D'Alembert reduction does not affect the polynomial degree of polynomial-hyperexponential solutions then a differential equation in the family (1) has a polynomial-hyperexponential solution of polynomial degree d if and only if so has any of its monic D'Alembert reductions. Therefore, if $\mathbb{L}_{2n,d}$ is a subvariety of $\mathbb{P}_{2n}$ then $\mathrm{codim}(\mathbb{L}_{2n,d}, \mathbb{P}_{2n}) = \mathrm{codim}(\mathbb{L}'_{2n,d}, \mathbb{M}_{2n})$.

Here we will analyze the existence of Liovillian solutions of equations in the family $\mathbb{M}_{2n}$. This is done in terms of some known theoretical results obtained by application of Kovacic's algorithm [12]. A first step is to introduce a system of coordinates in $\mathbb{M}_{2n}$ that fits our analysis of Eq. (3) better than the coefficients of $R(x)$. The following Lemma that can be traced back to [15, p. 474], allows to decompose the monic polynomial $R(x)$ in a suitable form for the application of the algorithm.

Lemma 3.1 *Every monic polynomial M(x) of even degree 2n can be written in one only way completing squares, that is,*

$$M(x) = A(x)^2 + B(x), \tag{5}$$

with $A(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$ *is a monic polynomial of degree n and* $B(x) = \sum_{j=0}^{n-1} b_j x^j$ *is a polynomial of degree at most* $n-1$*.*

According to the proof given in [1, Lemma 2.4, p. 275] it also clear that the decomposition map $\mathbb{M}_{2n} \to \mathbb{C}^{2n}$, $R(x) \mapsto (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ where $R(x) = A(x)^2 + B(x)$ is a regular invertible polynomial map. Therefore, we may consider the coefficients of $A(x)$ and $B(x)$ as a system of regular coordinates is $\mathbb{M}_{2n}$. The following results gives us precise information about the sets $\mathbb{L}'_{2n}$ and $\mathbb{L}'_{2n,d}$.

Theorem 3.2 [1, Theorem 2.5, pp. 276] *Let us consider the differential equation,*

$$y'' = M(x)y, \tag{6}$$

with $M(x) \in \mathbb{C}[x]$ *a monic polynomial of degree* $k > 0$*. Then its differential Galois Group G with coefficients in* $\mathbb{C}(x)$ *falls in one of the following cases:*

1. $G = SL_2(\mathbb{C})$ *(non-abelian, non-solvable, connected group).*
2. $G = \mathbb{C}^* \ltimes \mathbb{C}$ *(non-abelian, solvable, connected group).*

Furthermore, the second case is given if and only if the following conditions holds:

3. *M(x) has even degree* $k = 2n$*,*
4. *Writing* $M(x) = A(x)^2 + B(x)$ *as in Lemma 3.1, the quantity* $\pm b_{n-1} - n$ *is a non-negative even integer 2d,* $d \in \mathbb{Z}_{\geq 0}$*.*
5. *There exist a monic polynomial* P_d *of degree d satisfying at least one of the following differential equations,*

$$P_d'' + 2AP_d' - (B - A')P_d = 0, \tag{7}$$

$$P_d'' - 2AP_d' - (B + A')P_d = 0. \tag{8}$$

In such case, Liouvillian solutions are given by

$$y_1 = P_d e^{\int A dx}, y_2 = y_1 \int \frac{e^{-2\int A dx}}{P_d^2}, or \tag{9}$$

$$y_1 = P_d e^{-\int A dx}, y_2 = y_1 \int \frac{e^{2\int A dx}}{P_d^2}. \tag{10}$$

A careful read of Theorem 3.2 gives us the following.

Corollary 3.3 *The sets* $\mathbb{L}'_{2n}$ *and* $\mathbb{L}'_{2n,d}$ *in* $\mathbb{M}_{2n}$ *satisfy the following.*

1. $\mathbb{L}'_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}'_{2n,d}$.
2. $\mathbb{L}_{2n,d}$ *is contained in the hypersurface of* $\mathbb{M}_{2n}$ *of equation* $b_{n-1}^2 - (n+2d)^2 = 0$.

Therefore, the sets $\mathbb{L}_{2n}$ *and* $\mathbb{L}_{2n,d}$ *in* $\mathbb{P}_{2n}$ *satisfy* $\mathbb{L}_{2n} = \bigcup_{d=0}^{\infty} \mathbb{L}_{2n,d}$.

Proof

1. It is a consequence of the dichotomy of the Galois group. In case the group is not $SL_2(\mathbb{C})$ it leads to a polynomial-hyperexponential solution.
2. It is a direct consequence of point 2 in the second part of Theorem 3.2, The last statement of the corollary is a consequence of the point 1. and the fact the the reductions process from $\mathbb{P}_{2n}$ to $\mathbb{M}_{2n}$ preserves polynomial-hyperexponential solutions. □

3.3 Canonical equation

The following example:

$$y'' = (x^{2n} + \mu x^{n-1})y, \quad \mu \in \mathbb{C}. \tag{11}$$

that we refer to as *canonical equation* gives us some information about the non emptiness of the sets $\mathrm{L}_{2n,d}$ for large d. Due to theorem 3.2, if (11) has a Liouvillian solution, the parameter μ in the canonical coefficient $x^{2n} + \mu x^{n-1}$ is forced to be a discrete parameter that can be $\mu = 2d + n$ or either $\mu = -2d - n$, where d is a non-negative integer, which lead us to deal with two different equations,

$$y'' = (x^{2n} + (2d+n)x^{n-1})y, \mathit{or} \tag{12}$$

$$y'' = (x^{2n} - (2d+n)x^{n-1})y. \tag{13}$$

Proposition 3.4 *The differential equation* (12) *is integrable in the liouvillian sense if and only if,* $d = (n+1)k$ *or* $d = (n+1)k + 1$ *where k is a non-negative integer.*

Proof The differential equation (12), is transformed into the Whittaker differential equation,

$$\mathcal{W}'' = \left(\frac{1}{4} - \frac{\frac{-2d-n}{2n+2}}{z} + \frac{4(\frac{1}{2n+2})^2 - 1}{4z^2} \right) \mathcal{W}, \tag{14}$$

through the change of variables $z = \frac{2}{n+1}x^{n+1}$, $y = z^{-\frac{n}{2n+2}}\mathcal{W}$. Applying Martinet-Ramis theorem, see [13], we have that

$$\pm \frac{-2d-n}{2n+2} \pm \frac{1}{2n+2} = \frac{1}{2} + k, k \in \mathbb{Z}_{\geq 0},$$

which left only two posibilities, $d = (n+1)k$ or $d = (n+1)k + 1$. □

It is easy to see that the change of variables made in above proof also transform the Eq. (13) into a Whittaker equation. Nevertheless this new equation will have parameters $\kappa = \frac{2d+n}{2n+2}$ and $\mu = \frac{1}{2n+2}$. So via Martinet–Ramis theorem we can enunciate the following result analogous to the previous proposition.

Proposition 3.5 *The differential equation (13) is integrable in the liouvillian sense if and only if, $d = (n+1)k$ or $d = (n+1)k + 1$ where k is a non-negative integer.*

Moreover, the solutions to the Eq. (11) can be explicitly written as

$$\begin{aligned} y_{d,n}(x) &= P_{d,n}(x)e^{\frac{x^{n+1}}{n+1}}, if \mu = 2d+n, or \\ y_{d,n}(x) &= P_{d,n}(x)e^{-\frac{x^{n+1}}{n+1}}, if \mu = -(2d+n), \end{aligned} \tag{15}$$

where the polynomials $P_{d,n}$ can be find by a Frobenius-like method. Having said that, it is a tedious process. In any case, for $d = (n+1)k$ we have that

$$P_{d,n}(x) = x^d + \sum_{j=n+1}^{d} \zeta_j x^{d-j}, where \zeta_j = 0 for j \neq (n+1)m. \tag{16}$$

On the other hand, for $d = (n+1)k + 1$

$$P_{d,n}(x) = x^d + \sum_{j=n+2}^{d} \zeta_j x^{d+1-j}, where \zeta_j = 0 for j \neq (n+1)m + 1. \tag{17}$$

ζ_j	$j = (n+1)m,\ j = (n+1)m + 1$
$\mu = 2d + 2$	$\prod_{r=1}^{m} -\frac{(d+2-r(n+1))(d+3-r(n+1))}{-2(d+2-r(n+1)-n)-2d}$
$\mu = -2d - 2$	$\prod_{r=1}^{m} -\frac{(d+2-r(n+1))(d+3-r(n+1))}{2(d+2-r(n+1)-n)+2d}$

Corollary 3.6 *For any pair (n, d) of degrees with $d \equiv 0$ or $d \equiv 1$ mod $(n+1)$ there exist a monic polynomial M(x) of degree 2n such that the equation*

$$y'' = M(x)y \tag{18}$$

has a polynomial-hyperexponential solution of exponential degree $n+1$ *and polynomial degree* d*; therefore* $\mathbb{L}_{2n,d}$ *is non-empty.*

4 Analysis of auxiliary equations

We refer to Eqs. (7) and (8) as *auxiliary equations* for Eq. (6). As it is stated in Theorem 3.2 the existence of a Liouvillian solution of Eq. (6) depends of the existence of a polynomial solution of the auxiliary equations. In what follows we will show that conditions for the existence of a polynomial solution P_d of given degree is algebraic in the coefficients of $A(x)$ and $B(x)$, and therefore in the coefficients of $M(x)$.

4.1 Asymptotic iteration method

The asymptotic iteration method or AIM was introduced by Ciftci et al in [7] as a tool to solve homogeneous differential equations of the form

$$y'' = \ell_0 y' + r_0 y \tag{19}$$

where ℓ_0 and r_0 are smooth functions defined on a real interval. Nevertheless, the method is purely differential algebraic, so we can extent the result to differential rings of characteristic zero. By derivation of Eq. (19) we obtain a sequence of differential equations,

$$y^{(j+2)} = \ell_j y' + r_j y \tag{20}$$

where the sequences $\{\ell_j\}_{j\in\mathbb{N}}$ and $\{r_j\}_{j\in\mathbb{N}}$ are defined by the recurrence,

$$\ell_{j+1} = \ell_j' + r_j + \ell_0 \ell_j, \quad r_{j+1} = r_j' + r_0 \ell_j. \tag{21}$$

and the sequence of obstructions,

$$\delta_j = r_j \ell_{j-1} - \ell_j r_{j-1}.$$

We say that the AIM *stabilizes* at $p > 0$ if $\delta_p = 0$. The following statement is a differential algebraic translation of [16, Theorem 1].

Theorem 4.1 *Let* ℓ_0 *and* r_0 *be elements of a differential field* R *of characteristic zero. If there exist* $p > 0$ *such that*

$$\frac{r_p}{\ell_p} = \frac{r_{p-1}}{\ell_{p-1}} := \alpha, \tag{22}$$

then differential equation (19) *has general solution,*

$$y = u^{-1}(c_2 + c_1\beta), \quad c_1, c_2 \text{ arbitrary constants,} \tag{23}$$

in the extension $R\langle \alpha, u, u^{-1}, v, \beta\rangle$ where u, v, β are solutions of $u' = \alpha u$, $v' = \ell_0 v$ y $\beta' = u^2 v$, respectively.

Proof By derivation of Eq. (19) we obtain,

$$y^{(p+2)} = \ell_p y' + r_p y,$$

and from there

$$\log(y^{(p+1)})' = \frac{\ell_p\left(y' + \frac{r_p}{\ell_p}y\right)}{\ell_{p-1}\left(y' + \frac{r_{p-1}}{\ell_{p-1}}y\right)}.$$

If condition (22) is satisfied, then we have

$$(y^{(p+1)})' = \frac{\ell_p}{\ell_{p-1}} y^{(p+1)}.$$

On the other hand, from the recurrence, we have,

$$\frac{\ell_p}{\ell_{p-1}} = \log(\ell_{p-1})' + \alpha + \ell_0,$$

and replacing into the above equation we obtain,

$$(y^{(p+1)})' = (\log(\ell_{p-1})' + \alpha + \ell_0)y^{(p+1)}.$$

We have that $y^{(p+1)} = c_1\ell_{p-1}uv$ is a general solution for this equation and finally we obtain

$$y' + \alpha y = c_1 uv$$

that yields the general solution of the statement. □

The AIM method tests whether the auxiliary equations have polynomial solution. The following statement is a differential algebraic translation of [16, Theorem 2]. There is no difference in the proof, so we refer the reader to the original text.

Theorem 4.2 *Let ℓ_0, r_0 be elements in a differential field R of characteristic zero that contains $\mathbb{C}[x]$.*

(i) *If (19) has a polynomial solution of degree p, then $\delta_p = 0$*
(ii) If $\ell_p\ell_{p-1} \neq 0$ *and $\delta_p = 0$, then the differential equation (19) has a polynomial solution of degree at most p.*

4.2 Liouvillian solutions by means of AIM

Let us proceed to the AIM of auxiliary equations (7) and (8). For Eq. (7) we should start with $\ell_0^+ = -2A(x)$ and $r_0^+ = B(x) - A'(x)$. By the recurrence law (21) we have a sequence:

$$\begin{bmatrix} \ell_{p+1}^+ \\ r_{p+1}^+ \end{bmatrix} = \begin{bmatrix} \ell_p^+ \\ r_p^+ \end{bmatrix}' + \begin{bmatrix} -2A(x) & 1 \\ B(x) - A'(x) & 0 \end{bmatrix} \begin{bmatrix} \ell_p^+ \\ r_p^+ \end{bmatrix}$$

A condition for the existence of a polynomial solution of degree at most p of (7) is the vanishing of the polynomial $\delta_p^+ = r_p^+ \ell_{p-1}^+ - r_{p-1}^+ \ell_p^+$. We proceed analogously with Eq. (8) obtaining sequences of polynomials r_p^-, ℓ_p^- and δ_p^-.

In order to model this process, let us consider $\mathbb{Q}\{a, b\}$ the ring of differential polynomials in two differential variables a, b. We may consider the following $\mathbb{Q}$ -linear differential operator in the space of 2 by 2 matrices (Table 1).

$$\varphi : \mathrm{Mat}_{2\times 2}(\mathbb{Q}\{a,b\}) \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}\{a,b\}), \quad C \mapsto \varphi(C) = C' + \begin{bmatrix} -2a & 1 \\ b - a' & 0 \end{bmatrix} C$$

We consider the iterations of this map. If we give to the differential variables a, b the values of the polynomials $A(x)$ and $B(x)$ we obtain:

$$\left(\varphi^{p+1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right)(A(x), B(x)) = \begin{bmatrix} \ell_p & \ell_{p-1} \\ r_p & r_{p-1} \end{bmatrix}.$$

Let us define the sequence of universal differential polynomials,

$$\Delta_p = -\det\left(\varphi^{p+1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \in \mathbb{Q}\{a, b\}.$$

As we will see this sequence $\{\Delta_d\}_{d\in\mathbb{N}}$ of differential polynomials governs the Liouvillian integrability of Eq. (6) for any even degree $2n$ of $M(x)$.

Table 1 First values of the universal differential polynomials Δ_n

Δ_0	$b - a'$
Δ_1	$2a\left(a'' - b'\right) + 4ba' - 3(a')^2 - b^2$
Δ_2	$-9b^2a' - 15(a')^3 - 2(-3a''b' + 2(a^2(a^{(3)} - b'') + (a'')^2) + (b')^2)$
	$+b(-a^{(3)} + 2a(3b' - 5a'') + 23(a')^2 + b'') + a'(3a^{(3)} - 2a(7b' - 9a'') - 3b'') + b^3$
Δ_3	$2(2a'' + 2a(b - 4a') + 4a^3 - b')(a^{(4)} - 4a^2(b' - a'') + b(10a'' - 4b') + 10a'b' + 8a^3(b - a')$
	$+2a(-a^{(3)} - 14ba' + 12(a')^2 + b'' + 2b^2) - 16a'a'' - b^{(3)}) - (-5a^{(3)} + a(26a'' - 10b')$
	$+12a^2(b - 5a') - 10ba' + 21(a')^2 + 16a^4 + 3b'' + b^2)(-a^{(3)} - 2a(b' - a'') + 4a^2(b - a')$
	$-6b(a') + 5a'^2 + b'' + b^2)$

Theorem 4.3 *Equation* (6) *with* $M(x) = A(x)^2 + B(x)$ *has a polynomial-hyperexponential solution of polynomial degree d if and only if,*

$$b_{n-1}^2 = (n+2d)^2 \quad \text{and} \quad \Delta_d(A(x), B(x))\Delta_d(-A(x), B(x)) = 0.$$

Therefore $\mathbb{L}'_{2n,d}$ *is an algebraic subvariety of* $\mathbb{M}_{2n}$ *contained in the union of the irreducible hypersurfaces of equations:*

$$b_{n-1} = 2d + n, \quad -b_{n-1} = 2d + n.$$

Proof Note that, by definition of the sequence Δ_d we have $\delta_d^+ = \Delta_d(A(x), B(x))$. Analogously, the application of the AIM to Eq. (8) produces an obstruction, δ_d^-. Note that, because of the symmetry between Eqs. (7) and (8) $\delta_d^- = \Delta_d(-A(x), B(x))$.

We need only to check that $\ell_{d-1}^{\pm}\ell_d^{\pm} \neq 0$ for the auxiliar equations. This comes easily from the fact that $\ell_0^{\pm} = \pm 2A(x)$ is of bigger degree than $r_0 = B \pm A'$. Note that $\Delta_d(A(x), B(x))\Delta_d(-A(x), B(x))$ is a polynomial in $x, a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}$. Its coefficients as a polynomial in x are the algebraic equations of the restricted spectral variety $\mathbb{L}'_{2n,d}$ in $\mathbb{M}_{2n}$. □

Example 4.4 As a first example of AIM applications, let us consider an equation on $\mathbb{M}_2$

$$y'' = ((x + a_0)^2 + b_0)y. \tag{24}$$

An elementary traslation as $x \mapsto x + a_0$ reduces the determination of $\mathbb{L}'_2$ structure to an analysis of liouvillian-integrability conditions for quantum harmonic oscillator

$$y'' = (x^2 + b_0)y. \tag{25}$$

These conditions are $b_0^2 = (2d+1)^2$ and $\Delta_d(x, b_0)\Delta_d(-x, b_0) = 0$. It is easy to verify that $\Delta_d(x, b_0) = \Delta_d(-x, b_0) = 2^{d+1}\prod_{k=0}^{d} d - k$. Therefore,

$$\mathbb{L}'_{2,d} \;:\; (b_0 + 2d + 1)(b_0 - 2d - 1) = 0 \tag{26}$$

Let us note that for a given equation $\mathbb{M}_{2n}$, conditions $b_{n+1} = 2d + n$ and $b_{n+1} = -2d - n$ are mutually exclusive. In the first case, auxiliary equation (7) may have a polynomial solution but not (8). The opposite occurs in the second case. Therefore, we decompose the spectral variety $\mathbb{L}'_{2n,d}$ as the disjoint union of two components $\mathbb{L}'_{2n,d} = \mathbb{L}^+_{2n,d} \cup \mathbb{L}^-_{2n,d}$. The first component $\mathbb{L}^+_{2n,d}$ correspond to equations whose auxiliary equation (7) has a polynomial solution of degree d and the second component $\mathbb{L}^-_{2n,d}$ correspond to equations whose auxiliary equation (8) has a polynomial solution of degree d.

Definition 4.5 $\mathbb{L}'_{2n,d} = \mathbb{L}^+_{2n,d} \cup \mathbb{L}^-_{2n,d}$, where

$$\mathbb{L}^+_{2n,d} = \begin{cases} b_{n-1} = 2d + n \\ \Delta_d(A(x), B(x)) = 0, \end{cases}$$

and

$$\mathbb{L}^-_{2n,d} = \begin{cases} b_{n-1} = -2d - n \\ \Delta_d(-A(x), B(x)) = 0. \end{cases}$$

As in the previous Example 4.4 it is always possible to get rid of the coefficient a_{n-1} by means of a translation in the x axis. Therefore is convenient to consider the sets,

$$V^{\pm}_{2n,d} = \{a_{n-1} = 0\} \cap \mathbb{L}^{\pm}_{2n,d}.$$

whose equations are easier to describe. For instance, in $\mathbb{M}_4$ and $\mathbb{M}_6$ we restrict our analysis to equations of the forms:

$$y'' = ((x^2 + a_0)^2 + b_1 x + b_0)y. \tag{27}$$

and

$$y'' = ((x^3 + a_1 x + a_0)^2 + b_2 x^2 + b_1 x + b_0)y \tag{28}$$

respectively. The following calculations of the equations of $V^+_{2n,d}$ for $n = 2, 3$ and small values of d, in Tables 2 and 3 is performed by means of the universal differential polynomials Δ_d.

4.3 Codimension of the spectral variety

As the degree in x of the polynomials $\Delta_p(A(x), B(x))$ grows quickly with p and the degree of $M(x) = A(x)^2 + B(x)$ it seems that the sets $\mathbb{L}_{2n,p}$ are smaller as p grows. However, a direct analysis of the auxiliary equations allows us to bound the codimension of the spectral varieties $\mathbb{L}_{2n,d}$ in $\mathbb{M}_{2n}$. As we have seen before the algebraic

Table 2 Algebraic equations of restricted spectral varieties $V^+_{4,d} = \mathbb{L}^+_{4,d} \cap \{a_1 = 0\}$ for small values of d

$V^+_{4,0}$	$\begin{cases} b_1 = 2 \\ b_0 = 0 \end{cases}$
$V^+_{4,1}$	$\begin{cases} b_1 = 4 \\ b_0^2 + 4a_0 = 0 \end{cases}$
$V^+_{4,2}$	$\begin{cases} b_1 = 6 \\ b_0^3 + 16a_0 b_0 - 16 = 0 \end{cases}$
$V^+_{4,3}$	$\begin{cases} b_1 = 8 \\ b_0^4 + 40a_0 b_0^2 - 96b_0 + 144a_0^2 = 0 \end{cases}$
$V^+_{4,4}$	$\begin{cases} b_1 = 10 \\ b_0^5 + 80a_0 b_0^3 - 336b_0^2 + 1024a_0^2 b_0 - 3072a_0 = 0 \end{cases}$
$V^+_{4,5}$	$\begin{cases} b_1 = 12 \\ b_0^6 - 140a_0 b_0^4 + 896b_0^3 - 4144a_0^2 b_0^2 + 28160a_0 b_0 - 14400a_0^3 - 25600 = 0 \end{cases}$
$V^+_{4,6}$	$\begin{cases} b_1 = 14 \\ b_0^7 + 224a_0 b_0^5 - 2016b_0^4 + 12544a_0^2 b_0^3 - 142848a_0 b_0^2 + 147456a_0^3 b_0 + 288000b_0 - 884736a_0^2 = 0 \end{cases}$

Table 3 Algebraic equations of restricted spectral varieties $V^+_{6,d} = \mathbb{L}^+_{6,d} \cap \{a_1 = 0\}$ for small values of d

$V^+_{6,0}$	$\begin{cases} b_2 = 3 \\ b_1 = 0 \\ b_0 - a_1 = 0 \end{cases}$
$V^+_{6,1}$	$\begin{cases} b_2 = 5 \\ 2a_1b_1 - 8a_0 - 2b_0b_1 = 0 \\ -6a_1 - b_1^2 + 2b_0 = 0 \\ 4a_1b_0 - 2a_0b_1 - 3a_1^2 - b_0^2 = 0 \end{cases}$
$V^+_{6,2}$	$\begin{cases} b_2 = 7 \\ 23a_1^2b_0 - 9a_1b_0^2 - 14a_0a_1b_1 + 6a_0b_0b_1 - 15a_1^3 - 24a_1 + 32a_0^2 + b_0^3 - 2b_1^2 + 8b_0 = 0 \\ 9a_1^2b_1 - 12a_1b_0b_1 + 6a_0b_1^2 + 24a_0b_0 - 24a_0a_1 + 3b_0^2b_1 - 12b_1 = 0 \\ -3a_1b_1^2 + 36a_1b_0 + 24a_0b_1 - 30a_1^2 - 6b_0^2 + 3b_0b_1^2 - 48 = 0 \\ 22a_1b_1 - 32a_0 + b_1^3 - 6b_0b_1 = 0 \end{cases}$
$V^+_{6,3}$	$\begin{cases} b_2 = 9 \\ 176a_1^3b_0 - 86a_1^2b_0^2 - 116a_0a_1^2b_1 + 16a_1b_0^3 - 20a_1b_1^2 + 264a_1b_0 + 80a_0a_1b_0b_1 \\ \quad - 12a_0^2b_1^2 - 144a_0^2b_0 - 12a_0b_0^2b_1 + 120a_0b_1 - 105a_1^4 - 372a_1^2 + 432a_0^2a_1 \\ \quad - b_0^4 - 36b_0^2 + 8b_0b_1^2 - 288 = 0 \\ 60a_1^3b_1 - 92a_1^2b_0b_1 + 56a_0a_1b_1^2 + 192a_0a_1b_0 + 36a_1b_0^2b_1 + 96a_1b_1 - 48a_0b_0^2 \\ \quad - 24a_0b_0b_1^2 - 288a_0^2b_1 - 144a_0a_1^2 + 576a_0 + 8b_1^3 - 4b_0^3b_1 = 0 \\ \quad - 18a_1^2b_1^2 + 372a_1^2b_0 - 132a_1b_0^2 + 24a_1b_0b_1^2 - 72a_0a_1b_1 - 12a_0b_1^3 - 72a_0b_0b_1 \\ \quad - 252a_1^3 - 288a_1 + 12b_0^3 - 6b_0^2b_1^2 + 48b_1^2 + 288b_0 = 0 \\ 4a_1b_1^3 - 48a_0b_1^2 + 136a_1^2b_1 - 160a_1b_0b_1 + 288a_0b_0 - 864a_0a_1 - 4b_0b_1^3 + 24b_0^2b_1 \\ \quad + 192b_1 = 0 \\ -52a_1b_1^2 + 144a_0b_1 + 120a_1b_0 - 252a_1^2 - b_1^4 + 12b_0b_1^2 - 12b_0^2 - 288 = 0 \end{cases}$

equations for $\mathbb{L}_{2n,0}$ are well expressed by the obstruction $\Delta_0(a, b) = b - a'$, so henceforth we will consider $d > 0$.

Proposition 4.6 *If* $\mathbb{L}'_{2n,d}$ *is not empty, then* $\operatorname{codim}(\mathbb{L}'_{2n,d}, \mathbb{M}_{2n}) \leq n$.

Proof Now, let us suppose that $P_d = \sum_{k=0}^{d} p_k x^k$ is a solution to one of the following algebraic equations

$$P''_d \pm 2AP'_d - (B \mp A')P_d = 0 \tag{29}$$

where $A = x^n + \sum_{k=1}^{n} a_{n-k}x^{n-k}$ and $B = \sum_{k=1}^{n} b_{n-k}x^{n-k}$. Hence the coefficients of the polynomial

$$\sum_{k=2}^{d} k(k-1)p_k x^{k-2} \pm \left(2x^n + \sum_{k=1}^{n} 2a_{n-k}x^{n-k}\right)\left(\sum_{k=1}^{d} kp_k x^{k-1}\right) \\ -\left(\sum_{k=1}^{n} b_{n-k}x^{n-k} \mp nx^{n-1} + \sum_{k=1}^{n-1}(n-k)a_{n-k}x^{n-1-k}\right)\left(\sum_{k=0}^{d} p_k x^k\right) = 0 \tag{30}$$

in $\mathbb{C}[x]$ vanish. This give place to a system of equations which are sufficient conditions for the existence of P_d,

$$\begin{bmatrix} a_1 - b_0 & 2a_2 & 2 & 0 & \cdots & 0 & 0 \\ 2a_2 - b_1 & 3a_1 - b_0 & 4a_0 & 6 & \cdots & 0 & 0 \\ & & & & \ddots & * & * \\ 0 & 0 & 0 & 0 & \cdots & 2(d-1)+n-b_{n-1} & (2d+n-1)a_{n-1} - b_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2d+n-b_{n-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{bmatrix} = 0. \tag{31}$$

We will denote the coefficient matrix of the system (31) by $M^{\pm}_{d,n}(A,B)$. Note this matrix has size $(d+n) \times (d+1)$ and it also has the property

$$M^{\pm}_{d,n}(A,B) = \left[\begin{array}{ccc|c} & M^{\pm}_{d-1,n}(A,B) & & \begin{matrix} 0 \\ \vdots \\ * \end{matrix} \\ \hline 0 & \cdots & 0 & 2d+n \pm b_{n-1} \end{array}\right]. \tag{32}$$

Remark 4.7 As there is no solution P of degree less than d, then $\operatorname{rank}(M^{\pm}_{d-1,n}(A,B)) = d$.

In order to determinate the codimension of $\mathbb{L}'_{2n,d}$ around a point (A_0, B_0) we shall choose a suitable $d \times d$ submatrix D of $M^{\pm}_{d-1,n}(A_0,B_0)$ such that its determinant is different from zero. In addition, the vanishing of the determinants of the matrices set by adding one of the remaining n rows of $M^{\pm}_{d-1,n}(A_0,B_0)$ to D, generates n conditional equations which guarantees the existence of a non-trivial solution to (31). □

4.4 An example: case $n = 3$

As an useful example in order to illustrate further computes, specially for looking accurate spectral values on Schrödinger type problems, let us assume that $A(x) = x^3 + a_{3,1}x^2 + a_{3,2}x + a_{3,3}$ and $B(x) = b_{2,0}x^2 + b_{2,1}x + b_{2,2}$. So, the analysis on previous Sect. 4.3 for case $n = 3$ can be summarized with the following proposition.

Proposition 4.8 *A necessary condition for equation*

$$y'' - (2x^3 + 2a_{3,1}x^2 + 2a_{3,2}x + 2a_{3,3})y' - ((b_{2,0}+3)x^2 + (2a_{3,1} + b_{2,1})x + a_{3,2} + b_{2,2})y = 0 \tag{33}$$

in order to have a polynomial solution of degree d is $b_{2,0} + 3 = -2d$ for $d = 0, 1, 2, \ldots$

On the other hand, sufficient conditions are coded by the solutions of the linear system associated to the matrix

$$M_d^-(A,B) = \begin{bmatrix} \alpha_0 & \beta_0 & \gamma_0 & & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & & & \eta_d & \zeta_d & \alpha_d \\ & & & & 0 & \eta_{d+1} & \zeta_{d+1} \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} \alpha_k &= -a_{3,2}(2k+1) - b_{2,2}, k = 0,1,2,\dots \\ \beta_k &= -2a_{3,3}(k+1), \qquad k = 0,1,2,\dots \\ \gamma_k &= (k+2)(k+1), \qquad k = 0,1,2,\dots \\ \zeta_k &= -2a_{3,1}k - b_{2,1}, \qquad k = 1,2,3\dots \\ \eta_k &= -2k - b_{2,0} + 1, \qquad k = 2,3,4\dots \end{aligned}$$

It creates a set of at most two polynomial equations in the variables $a_{3,0}$, $a_{3,1}$, $a_{3,2}$, $a_{3,3}$, $a_{2,0}$, $a_{2,1}$, $a_{2,2}$ which guarantees likewise a non-trivial solution to the associated system to $M_d^-(A,B)$ and a polynomial solution to Eq. (33).

Proof This is a restriction of the analysis developed on Sect. 4.3 to $n = 3$. $\square$

Generically we can suppose that our $d \times d$ principal minor is different from zero, so the equations given by Proposition 4.8 are the determinants

$$\Delta^1_{d,3}(-A,B) = \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & \eta_d & \zeta_d & \alpha_d & \end{vmatrix} = 0 \tag{35}$$

and

$$\Delta^2_{d,3}(-A,B) = \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & & & & \\ \zeta_1 & \alpha_1 & \beta_1 & \gamma_1 & & & \\ \eta_2 & \zeta_2 & \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \eta_{d-2} & \zeta_{d-2} & \alpha_{d-2} & \beta_{d-2} & \gamma_{d-2} \\ & & \eta_{d-1} & \zeta_{d-1} & \alpha_{d-1} & \beta_{d-1} \\ & & 0 & \eta_{d+1} & \zeta_{d+1} & \end{vmatrix} = 0 \tag{36}$$

Several detailed examples of this equations can be found on [8, 19].

4.5 Proof of Theorem 2.3

We can now state the proof, which follows easily from the other results. Statement (a) is a direct consequence of Proposition 3.5. Statement (b) is a consequence of Theorems 4.3 and Proposition 4.6. Note that, from d'Alembert reduction, the codimension of $\mathbb{L}'_{2n,d}$ in $\mathbb{M}_{2n}$ coincide with that of $\mathbb{L}_{2n,d}$ in $\mathbb{P}_{2n}$. Statement (c) and (d) are also clear, as $\mathbb{L}'_{2n,d}$ is contained in the union of hyperplanes of equations $b_{n+1} = 2d + n$ and $b_{n-1} = -2d - n$. □

5 Schrödinger equation

Let us summarize briefly the known results about explicit solutions for the one dimensional stationary Schrödinger equation. We start mentioning that Natanzon in 1971, see [14], introduced *exactly solvable potentials*, which today are known as *Natanzon potentials*. The seminal work of Natanzon inspired further researchers about exactly solvable potentials, although in the sense of Natanzon exactly solvable potentials also include potentials in where Schrödinger equations have eigenfunctions of hypergeometric type, not necessarily Liouvillian functions. The exactly solvable potentials, also known as solvable potentials, we extended to Schrödinger equations with explicit eigenfunctions. In this sense, solvable potentials are related to Schrödinger equations with eigenfunctions belonging to the set of special functions (Airy, Bessel, Error, Ei, Hypergeometric, Whittaker, Heun), not necessarily Liouvillian! Moreover, in case of Coulomb and 3D harmonic oscillator potentials correspond to Schrödinger equations which are transformed into Whittaker differential equations, Martinet-Ramis in [13] established the necessary and sufficient conditions to determine the obtaining of Liouvillian solutions of the Whittaker differential equations. Recently Combot in [9] developed another method to obtain exactly solvable potentials, in the sense of Natanzon, involving *rigid functions* in the sense of Katz.

To avoid confusion between explicit and Liouvillian solutions it was introduced the concept of *algebraic spectrum* in [2]. Also known as Liouvillian spectral set it is the set of eigenvalues for which the Schrödinger equation has Liouvillian eigenfunctions, see also [3, 4]. In some scenarios it is known that bounded eigenfunctions of Schrödinger operator are necessarily Liouvillian, see [6]. Potentials with infinite countable algebraic spectrum are called *algebraically solvable potentials* and those with finite algebraic spectrum *algebraically quasi-solvable potentials*, for complete details see [4, §3.1, pp. 316] and see also [2, 3].

On the other hand, Turbiner in 1988, see [18], following the same philosophy of Natanzon, introduced *quasi-solvable potentials*. The seminal paper of Turbiner leaded to the seminal paper of Bender and Dunne in 1996, see [5], in where they obtain a family of orthogonal polynomials in the energy values of the Schrödinger equation with sextic anharmonic potentials, see also [11] for the study of more general sextic anharmonic oscillators. Due to Schrödinger equation with quartic anharmonic oscillator potential falls in *triconfluent Heun equation*, see [10], it is in some sense a generalized Natanzon potential (exactly solvable) although there no exist Liouvillian eigenfunctions. In a similar way for algebraically solvable potentials, in [4, §3.1, pp. 316] also was introduced the concept of *algebraically quasi-solvable potential* as those finite non empty algebraic spectrum, see also [2, 3]. Examples of algebraically solvable potentials and algebraically quasi-solvable potentials (quartic and sextic oscillators) were presented in [2–4] using [1, Theorem 2.5, pp. 276], which corresponds to the application of Kovacic algorithm for reduced second order linear differential equation with polynomial coefficients.

Let us consider the one dimensional stationary Schrödinger equation

$$\psi'' = (\lambda - U(x))\psi \tag{37}$$

with a polynomial potential $U(x)$. It is clear that the potential $U(x)$ is *algebraically quasi-exactly solvable* if there are some values of λ for wich equation (37) has a Liouvillian solution. This is equivalent to say that the line,

$$\{\lambda - U(x) \ : \ \lambda \in \mathbb{C}\} \subseteq \mathbb{M}_{2n}$$

paremeterized by λ, intersects the spectral set $\mathbb{L}_{2n}$.

As it is well know, and we examined in Example 4.4, any quadratic potential is quasi-exactly solvable (and more over, exactly solvable). It is also clear that any quasi-exactly solvable potential is of even degree. Let us assume from now on that $U(x)$ is of degree $2n \geq 4$.

We consider the decomposition $-U(x) = A(x)^2 + B(x)$ as in Theorem 3.2. We define the *arithmetic condition* of $U(x)$ as the complex number,

$$d = \frac{|b_{n-1}| - n}{2}$$

where b_{n-1} is the coefficient of x^{n-1} the polynomial $B(x)$ appearing in the unique decomposition $-U(x) = A(x)^2 + B(x)$. Note that a necessary condition for $U(x)$ to be quasi exactly solvable is its arithmetic condition to be a non-negative integer. In such case the intersection between the line:

$$\{\lambda - U(x) \ : \ \lambda \in \mathbb{C}\} \subseteq \mathbb{M}_{2n}$$

and $\mathbb{L}_{2n}$ is confined to the spectral variety $\mathbb{L}_{2n,d}$.

Let us consider the universal sequence of differential polynomials $\Delta_d \in \mathbb{Q}\{a, b\}$ as in Theorem 4.3. The following lemma allows us to bound the number of admissible values of energy (for which the Schrödinger equation admits a Liouvillian solution) of any quasi-exactly solvable polynomial potential. Let us make clear that by the degree of a differential polynomial Δ_d in the variable b we mean its ordinary degree: that is we consider $a, a', a'', \ldots, b, b', b'', \ldots$ as an infinite set of independent variables.

Lemma 5.1 *The degree of Δ_d in the variable b is at most $d + 1$.*

Proof Let us recall the differential polynomials ℓ_d and r_d appearing in the definition of Δ_d. Let us prove first:

(a) The degree of ℓ_d in the variable b is small or equal to $\frac{d+1}{2}$.
(b) The degree of r_d in the variable b is small or equal to $\frac{d+2}{2}$.

The degree of $\ell_0 = -2a$ in the variable b is 0 an the degree of $r_0 = b - a'$ in the variable b is 1. Therefore (a) and (b) hold for $d = 0$. Now, from the recurrence law (21) we have that the degree in b of ℓ_{j+1} is at most that of r_j and that the degree in b of r_{j+1} is at most a unit bigger that the degree of ℓ_j. This proves (a) and (b). The degree of δ_d is at most the maximum between the sum of the degrees of ℓ_d and r_{d-1} and the sum of the degrees of ℓ_{d-1} and r_d; which is at most $d + 1$. □

Theorem 5.2 *Let $U(x)$ be an algebraically quasi-solvable polynomial potential, and let d be its arithmetic condition. The number of values of the energy parameter λ such that Eq.* (37) *has a Liouvillian solution is at most $d + 1$.*

Proof Generically, we may consider that $U(x)$ has no independent term. Then the condition on λ for the existence of a Liouvillian solution is the vanishing of $\Delta_d(A(x), B(x) + \lambda)$ which is a polynomial in x of λ. The number of values of λ for which this polynomial vanish can not be greater than its degree in λ. Clearly, the degree in λ of $\Delta_d(A(x), B(x) + \lambda)$ can not exceed the degree in b of $\Delta_d(a, b)$ which is bounded by $d + 1$ by Lemma 5.1. □

Example 5.3 In order to illustrate the procedures developed here let us consider the non-singular Turbiner potential

Table 4 Spectral system of Schrödinger equation associated to (38)

d	$\mathbf{C}_d$
1	$\begin{cases} J = 1 \\ \lambda = 0 \end{cases}$
3	$\begin{cases} J = 2 \\ \lambda^2 - 24 = 0 \end{cases}$
5	$\begin{cases} J = 3 \\ \lambda^3 - 128\lambda = 0 \end{cases}$
7	$\begin{cases} J = 4 \\ \lambda^4 - 400\lambda^2 + 12096 = 0 \end{cases}$
9	$\begin{cases} J = 5 \\ -\lambda^5 + 960\lambda^3 - 129024\lambda = 0 \end{cases}$
11	$\begin{cases} J = 6 \\ \lambda^6 - 1960\lambda^4 + 729280\lambda^2 - 26611200 = 0 \end{cases}$
13	$\begin{cases} J = 7 \\ -\lambda^7 + 3584\lambda^5 - 2934784\lambda^3 + 438829056\lambda = 0 \end{cases}$

$$U(x) = x^6 - (4J + 1)x^2 \tag{38}$$

where J is a non-negative integer. This potential has been studied in several papers, including [5]. Let $\mathbf{C}_d \subset \mathbb{L}'_{6,d}$ be the set consisting of all possible values for J and λ with polynomial hyperexponential solutions of polynomial degree d. In virtue of Theorem 4.3 it is a subvariety of $V(2J - d - 1)$. So, d shall only take non-negative odd values (Table 4).

On the other hand, we can easily compute the equations of $\mathbf{C}_d$ through the universal differential polynomial $\Delta_d(x^3, -(4J + 1)x^2 - \lambda)$ for the auxiliary equation

$$P''_d - 2x^3 P'_d - (3x^2 - (4J + 1)x^2 - \lambda)P_d = 0. \tag{39}$$

For the case $d = 1$ we get the following equations

$$\begin{cases} 2J - 2 = 0 \\ -\lambda^2 = 0 \\ 2(-4J - 1)\lambda + 12\lambda = 0 \\ -(-4J - 1)^2 - 8(-4J - 1) - 15 = 0. \end{cases} \tag{40}$$

Taking into account above consideration we compute the first seven equations for $\mathbf{C}_d$

6 Final remarks

In this paper we developed a technique to obtain Liouvillian solutions for parameterized second order linear differential equations with polynomial coefficients. In particular case, we study the set of possible values of energy to get Liouvillian solutions of Schrödinger equations with anharmonic potentials. We adapted asymptotic iteration method, Kovacic's Algorithm and previous results provided in [1–4] in terms of algebraic varieties extending slightly the known results about polynomial quasi-solvable potentials.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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