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A Posterior Ensemble Kalman Filter Based On A Modified Cholesky Decomposition

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Abstract

In this paper, we propose a posterior ensemble Kalman filter (EnKF) based on a modified Cholesky decomposition. The main idea behind our approach is to estimate the moments of the analysis distribution based on an ensemble of model realizations. The method proceeds as follows: initially, an estimate of the precision background error covariance matrix is computed via a modified Cholesky decomposition and then, based on rank-one updates, the Cholesky factors of the inverse background error covariance matrix are updated in order to obtain an estimate of the inverse analysis covariance matrix. The special structure of the Cholesky factors can be exploited in order to obtain a matrix-free implementation of the EnKF. Once the analysis covariance matrix is estimated, the posterior mode of the distribution can be approximated and samples about it are taken in order to build the posterior ensemble. Experimental tests are performed making use of the Lorenz 96 model in order to assess the accuracy of the proposed implementation. The results reveal that, the accuracy of the proposed implementation is similar to that of the well-known local ensemble transform Kalman filter and even more, the use of our estimator reduces the impact of sampling errors during the assimilation of observations.

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Keywords: Ensemble Kalman Filter, Posterior Ensemble, Modified Cholesky Decomposition

1 Introduction

To be concise, Data Assimilation is the process by which imperfect numerical forecasts are adjusted according real noisy observations [1]. In practice, Gaussian errors are commonly assumed for background and observational errors during the assimilation of observations [2]. While observational errors can be well-estimated in the context of operational data assimilation, background error correlations can be hard to approximate, mainly, owing to the size of the vector state which, typically, ranges in the order of millions [3]. In the ensemble Kalman filter (EnKF), an ensemble of model realizations is utilized in order to estimate the moments of the underlying background error distribution [4]. Since the ensemble size is constrained by

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computational aspects, localization methods can be utilized in order to mitigate the impact of sampling errors [5, 6]. Efficient formulations of the EnKF account for some sort of implicit localization during the analysis step in order to damp out spurious correlations [7, 8, 9]. For instance, in the EnKF based on a modified Cholesky decomposition [10], a band estimate of the inverse background error covariance matrix can be obtained, on the fly, during the assimilation of observations. Even more, this precision matrix can be expressed in terms of Cholesky factors which are composed by a diagonal matrix and a band lower triangular matrix. We think that, these factors can be updated in order to estimate the Cholesky factors of the inverse analysis covariance matrix. This covariance matrix can be estimated by applying a series of rank-one updates on the Cholesky factors of the inverse background error covariance matrix. With such covariance matrix, samples from the posterior error distribution can be approximately taken with low-computational efforts.

This paper is organized as follows: in section 2 efficient EnKF formulation are discussed, section 3 presents the proposed EnkF implementation, in section 4 the accuracy of the proposed EnKF is assessed and compared with well-known EnKF formulations and finally, conclusions are stated in section 5.

2 Preliminaries

In this section, efficient EnKF implementations in order to avoid the impact of sampling errors on the analysis innovations are discussed. Inflation aspects and other sources of misestimation of model states and ensemble collapsing are well-studied in [11, 12].

In the ensemble Kalman filter, an ensemble of model realizations,

$$\mathbf{X}^b = \left[\mathbf{x}^{b[1]}, \mathbf{x}^{b[2]}, \ldots, \mathbf{x}^{b[N]}\right] \in \mathbb{R}^{n \times N}, \tag{1}$$

is utilized in order to estimate, the moments of the background error distribution,

$$\mathbf{x} \sim \mathcal{N}\left(\mathbf{x}^b, \mathbf{B}\right),$$

via the empirical moments of the ensemble (1), therefore,

$$\mathbf{x}^b \approx \overline{\mathbf{x}}^b = \frac{1}{N} \cdot \sum_{i=1}^{N} \mathbf{x}^{b[i]} \in \mathbb{R}^{n \times 1}, \tag{2a}$$

and

$$\mathbf{B} \approx \mathbf{P}^b = \frac{1}{N-1} \cdot \mathbf{\Delta X} \cdot \mathbf{\Delta X}^T \in \mathbb{R}^{n \times n}, \tag{2b}$$

where n is the model dimension, N is the ensemble size, $\mathbf{x}^{b[i]} \in \mathbb{R}^{n \times 1}$ is the i-th ensemble member, for $1 \leq i \leq N$, $\mathbf{x}^b \in \mathbb{R}^{n \times 1}$ is well-known as the background state while $\mathbf{B} \in \mathbb{R}^{n \times n}$ stands for background error covariance matrix, $\overline{\mathbf{x}}^b$ is the ensemble mean, and $\mathbf{P}^b$ is the ensemble covariance matrix. Likewise, the matrix of member deviations $\mathbf{\Delta X} \in \mathbb{R}^{n \times N}$ reads,

$$\mathbf{\Delta X} = \mathbf{X}^b - \overline{\mathbf{x}}^b \cdot \mathbf{1}_N^T. \tag{3}$$

When an observation $\mathbf{y} \in \mathbb{R}^{m \times 1}$ is available, the analysis ensemble can be computed as follows,

$$\mathbf{X}^a = \mathbf{X}^b + \mathbf{Z} \in \mathbb{R}^{n \times N}, \tag{4}$$

where $\mathbf{Z} \in \mathbb{R}^{n \times N}$ can be obtained by the solution of the linear system of equations,

$$\left[\left[\mathbf{P}^b\right]^{-1} + \mathbf{H}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{H}\right] \cdot \mathbf{Z} = \mathbf{H}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{\Delta Y}\,, \tag{5}$$

$\mathbf{H} \in \mathbb{R}^{m \times n}$ is a linearized observational operator, $\mathbf{R} \in \mathbb{R}^{m \times m}$ is the estimated data-error covariance matrix, the matrix of innovations on the observations $\mathbf{\Delta Y} \in \mathbb{R}^{m \times N}$ reads,

$$\mathbf{\Delta Y} = \mathbf{y} \cdot \mathbf{1}_N^T + \mathbf{E} - \mathbf{H} \cdot \mathbf{X}^b\,, \tag{6}$$

and the columns of $\mathbf{E} \in \mathbb{R}^{m \times N}$ are samples from a zero-mean Normal distribution with covariance matrix $\mathbf{R}$. The forecast is the approximated by propagating the ensemble (4) until new observations are available,

$$\mathbf{x}_{\text{next}}^{b[i]} = \mathcal{M}_{t_{\text{current}} \rightarrow t_{\text{next}}}\left(\mathbf{x}_{\text{current}}^{a[i]}\right)\,, \text{ for } 1 \le i \le N\,, \tag{7}$$

where $\mathbf{x}^{a[i]} \in \mathbb{R}^{n \times 1}$ denotes the i-th analysis member, and $\mathcal{M}_{\text{current} \rightarrow \text{next}} : \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{n \times 1}$ is an imperfect numerical model (i.e., a model which mimics the behaviour of the ocean and/or atmosphere).

In operational data assimilation, ensemble members can come at high computational efforts [13] and therefore, the ensemble moments (2) are corrupted by sampling noise [14]. Hence, localization methods are commonly utilized in the EnKF context in order to mitigate the impact of sampling errors. One of the best EnKF implementations is the local ensemble transform Kalman filter (LETKF) [15, 16]. In the LETKF, the analysis is approximated in the ensemble space,

$$\overline{\mathbf{x}}^a = \overline{\mathbf{x}}^b + \mathbf{\Delta X} \cdot \mathbf{a}^a \in \mathbb{R}^{n \times 1}\,,$$

where,

$$\mathbf{a}^a = \mathbf{Q} \cdot \mathbf{V}^T \cdot \mathbf{R}^{-1} \cdot \left[\mathbf{y} - \mathbf{H} \cdot \overline{\mathbf{x}}^b\right] \in \mathbb{R}^{N \times 1}\,, \tag{8}$$

$\mathbf{V} = \mathbf{H} \cdot \mathbf{\Delta X} \in \mathbb{R}^{m \times N}$, and an estimate of the analysis covariance matrix in such space reads,

$$\mathbf{Q} = \left[(N-1) \cdot \mathbf{I} + \mathbf{V}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{V}\right]^{-1} \in \mathbb{R}^{N \times N}\,,$$

with $\mathbf{I} \in \mathbb{R}^{N \times N}$ being the identity matrix in the ensemble space. This covariance matrix can then be utilized in order to build an ensemble about the posterior model of the distribution. In this context, localization methods are performed by using domain decomposition [17]: each model component is surrounded by a local box of radius r and only local information (i.e., observed components) are utilized during the assimilation step.

In the ensemble Kalman filter based on a modified Cholesky decomposition (EnKF-MC) [18] background error correlations are estimated via the Cholesky decomposition proposed by Bicket and Levina in [19]. This provides an estimate of the inverse background error covariance matrix of the form,

$$\widehat{\mathbf{B}}^{-1} = \mathbf{L}^T \cdot \mathbf{D} \cdot \mathbf{L} \in \mathbb{R}^{n \times n}\,, \tag{9}$$

where $\mathbf{L} \in \mathbb{R}^{n \times n}$ is an unitary lower-triangular matrix, and $\mathbf{D} \in \mathbb{R}^{n \times n}$ is a diagonal matrix. Even more, when only local effects are considered during the estimation of $\widehat{\mathbf{B}}^{-1}$, in addition, the matrix $\mathbf{L}$ is sparse with only a few non-zero elements per row. Typically, the number of non-zero elements are some function of the radius of influence during the estimation of background error correlations. For instance, in the one-dimensional case, the radius of influence denotes the maximum number of non-zero elements, per row, in $\mathbf{L}$. The EnkF-MC is then obtained by replacing in the estimator (9) in (4). Given the structure of the Cholesky factors, the EnKF-MC can be seen as a matrix-free implementation of the EnKF.

Recall that, the precision analysis covariance matrix reads,

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} + \mathbf{H}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{H} \in \mathbb{R}^{n \times n}\,. \tag{10}$$

and since $\mathbf{H}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{H} \in \mathbb{R}^{n \times n}$ can be written as a sum of m rank-one matrices, the factors (9) can be updated in order to obtain an estimate of the inverse analysis covariance matrix. In the next section, we propose an ensemble Kalman filter implementation based on this idea.

3 Proposed Method

Before we start, we make the assumptions [8, 20] that, in practice, the data error covariance matrix $\mathbf{R}$ has a simple structure, the observation operator $\mathbf{H}$ is sparse and therefore, it can be applied efficiently, and that the number of model components n is several times the ensemble size N. We want to estimate the moments of the analysis distribution,

$$\mathbf{x} \sim \mathcal{N}\left(\mathbf{x}^a,\, \mathbf{A}\right)\,,$$

based on the background ensemble (1), where $\mathbf{x}^a$ is the analysis state and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the analysis covariance matrix. Consider the estimate of the inverse background error covariance matrix (9), the precision analysis covariance matrix (10) can be approximated as follows,

$$\mathbf{A}^{-1} \approx \widehat{\mathbf{A}}^{-1} = \widehat{\mathbf{B}}^{-1} + \mathbf{X} \cdot \mathbf{X}^T\,, \tag{11}$$

where $\mathbf{X} = \mathbf{H}^T \cdot \mathbf{R}^{-1/2} \in \mathbb{R}^{n \times m}$. The matrix (11) can be written as follows,

$$\widehat{\mathbf{A}}^{-1} = \mathbf{L}^T \cdot \mathbf{D} \cdot \mathbf{L} + \sum_{i=1}^{m} \mathbf{x}_i \cdot \mathbf{x}_i^T\,,$$

where $\mathbf{x}_i$ denotes the i-th column of matrix $\mathbf{X}$, for $1 \le i \le m$. Consider the sequence of factors updates

$$\begin{aligned}
{\mathbf{L}^{(i)}}^T \cdot \mathbf{D}^{(i)} \cdot \mathbf{L}^{(i)} &= \left[\mathbf{L}^{(i-1)}\right]^T \cdot \mathbf{D}^{(i-1)} \cdot \mathbf{L}^{(i-1)} + \mathbf{x}_i \cdot \mathbf{x}_i^T \\
&= \left[\mathbf{L}^{(i-1)}\right]^T \cdot \left[\mathbf{D}^{(i-1)} + \mathbf{p}_i \cdot \mathbf{p}_i^T\right] \cdot \mathbf{L}^{(i-1)} \\
&= \left[\tilde{\mathbf{L}}^{(i-1)} \cdot \mathbf{L}^{(i-1)}\right]^T \cdot \tilde{\mathbf{D}}^{(i-1)} \cdot \left[\tilde{\mathbf{L}}^{(i-1)} \cdot \mathbf{L}^{(i-1)}\right]\,,
\end{aligned}$$

where $\mathbf{L}^{(i-1)} \cdot \mathbf{p}_i = \mathbf{x}_i \in \mathbb{R}^{n \times 1}$, for $1 \le i \le m$, $\widehat{\mathbf{B}}^{-1} = \left[\mathbf{L}^{(0)}\right]^T \cdot \mathbf{D}^{(0)} \cdot \mathbf{L}^{(0)}$, and

$$\mathbf{D}^{(i-1)} + \mathbf{p}_i \cdot \mathbf{p}_i^T = \left[\tilde{\mathbf{L}}^{(i)}\right]^T \cdot \tilde{\mathbf{D}}^{(i)} \cdot \tilde{\mathbf{L}}^{(i)} \in \mathbb{R}^{n \times n}\,. \tag{12}$$

We can make use of the Dolittle's method in order to compute the factors $\tilde{\mathbf{D}}^{(i)}$ and $\tilde{\mathbf{L}}^{(i)}$ in (12), it is enough to note that,

$$\underbrace{\begin{bmatrix} 1 & \tilde{l}_{21} & \tilde{l}_{31} & \dots & \tilde{l}_{n1} \\ 0 & 1 & \tilde{l}_{32} & \dots & 0 \\ 0 & 0 & 1 & \dots & \tilde{l}_{n2} \\ \vdots & \vdots & \vdots & \ddots & \tilde{l}_{n3} \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}}_{\tilde{\mathbf{L}}^{(i)\,T}} \cdot \underbrace{\begin{bmatrix} \tilde{d}_1 & 0 & 0 & \dots & 0 \\ 0 & \tilde{d}_2 & 0 & \dots & 0 \\ 0 & 0 & \tilde{d}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}}_{\tilde{\mathbf{D}}^{(i)}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \tilde{l}_{21} & 1 & 0 & \dots & 0 \\ \tilde{l}_{31} & \tilde{l}_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \tilde{l}_{n1} & \tilde{l}_{n2} & \tilde{l}_{n3} & \dots & 1 \end{bmatrix}}_{\tilde{\mathbf{L}}^{(i)}}$$

$$= \underbrace{\begin{bmatrix} d_1 + p_1^2 & p_1 \cdot p_2 & p_1 \cdot p_3 & \dots & p_1 \cdot p_n \\ p_2 \cdot p_1 & d_2 + p_2^2 & p_2 \cdot p_3 & \dots & p_2 \cdot p_n \\ p_3 \cdot p_1 & p_3 \cdot p_2 & d_3 + p_3^2 & \dots & p_3 \cdot p_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n \cdot p_1 & p_n \cdot p_2 & p_n \cdot p_3 & \dots & d_n + p_n^2 \end{bmatrix}}_{\mathbf{D}^{(i)} + \mathbf{p}_i \cdot \mathbf{p}_i^T} .$$

After some math simplifications, the next equations are obtained,

$$\tilde{d}_k = p_k^2 + d_k - \sum_{q=k+1}^{n} \tilde{d}_q \cdot \tilde{l}_{qi}^2 \,, \tag{13a}$$

and

$$\tilde{l}_{kj} = \frac{1}{\tilde{d}_k} \cdot \left[p_k \cdot p_j - \sum_{q=k+1}^{n} \tilde{d}_q \cdot \tilde{l}_{qi} \cdot \tilde{l}_{qj} \right] , \tag{13b}$$

for $1 \leq k \leq n$, and $1 \leq j \leq k-1$. The set of equations (13) can be used in order to derive an algorithm for rank-one update of Cholesky factors, the updating process is shown in the Algorithm 1.

Algorithm 1 Rank-one update for the factors $\mathbf{L}^{(i-1)}$ and $\mathbf{D}^{(i-1)}$.

```
function UPD_CHOLESKY_FACTORS(L^(i−1), D^(i−1), x_i)
    Compute p_i from L^(i)T · p_i = x_i.
    for k = n → 1 do
        Compute d̃_k via equation (13a).
        Set l_kk ← 1.
        for j = 1 → k − 1 do
            Compute l̃_kj according to (13b).
        end for
    end for
    Set L^(i) ← L̃^(i−1) · L^(i−1) and D^(i) ← D̃^(i).
    return L^(i), D^(i)
end function
```

Algorithm 1 can be used in order to update the factors of $\widehat{\mathbf{B}}^{-1}$ for all column vectors in $\mathbf{X}$, this process is detailed in the Algorithm 2. Once the updating process has been performed, the

Algorithm 2 Computing the factors $\mathbf{L}^{(m)}$ and $\mathbf{D}^{(m)}$ of $\widehat{\mathbf{A}}^{-1} = \mathbf{L}^{(m)\,T} \cdot \mathbf{D}^{(m)} \cdot \mathbf{L}^{(m)}$.

```
function COMPUTE_ANALYSIS_FACTORS(L^(0), D^(0), H, R)
    Set X ← H^T · R^{-1/2}.
    for i = 1 → m do
        Set [L^(i), D^(i)] ← UPD_CHOLESKY_FACTORS (L^(i-1), D^(i-1), x_i)
    end for
    return L^(m), D^(m)
end function
```

resulting factors form an estimate of the inverse analysis covariance matrix,

$$\widehat{\mathbf{A}}^{-1} = \left[\mathbf{L}^{(m)}\right]^T \cdot \mathbf{D}^{(m)} \cdot \mathbf{L}^{(m)} \in \mathbb{R}^{n \times n}\,. \tag{14a}$$

From this covariance matrix, the posterior mode of the distribution can be approximated as follows,

$$\overline{\mathbf{x}}^a = \overline{\mathbf{x}}^b + \mathbf{z} \in \mathbb{R}^{n \times 1}\,, \tag{14b}$$

where

$$\left[\mathbf{L}^{(m)}\right]^T \cdot \mathbf{D}^{(m)} \cdot \mathbf{L}^{(m)} \cdot \mathbf{z} = \mathbf{q}\,, \tag{14c}$$

with $\mathbf{q} = \mathbf{H}^T \cdot \mathbf{R}^{-1} \cdot \left[\mathbf{y} - \mathbf{H} \cdot \overline{\mathbf{x}}^b\right] \in \mathbb{R}^{n \times 1}$. Notice, the linear system (14c) involves lower and upper triangular matrices and therefore, $\overline{\mathbf{x}}^a$ can be estimated without the needing of matrix inversion. Once the posterior mode is computed, the analysis ensemble is built about it. Note that, $\widehat{\mathbf{A}}$ reads,

$$\widehat{\mathbf{A}} = \left[\mathbf{L}^{(m)}\right]^{-1} \cdot \left[\mathbf{D}^{(m)}\right]^{-1} \cdot \left[\mathbf{L}^{(m)}\right]^{-T}\,,$$

and therefore a square root of $\widehat{\mathbf{A}}$ can be approximated as follows,

$$\widehat{\mathbf{A}}^{1/2} = \left[\mathbf{L}^{(m)}\right]^{-1} \cdot \left[\mathbf{D}^{(m)}\right]^{-1/2} \in \mathbb{R}^{n \times n}\,, \tag{15}$$

which can be utilized in order to build the analysis ensemble,

$$\mathbf{X}^a = \overline{\mathbf{x}}^a \cdot \mathbf{1}_N^T + \boldsymbol{\Delta}\mathbf{X}^a\,, \tag{16}$$

where $\boldsymbol{\Delta}\mathbf{X}^a \in \mathbb{R}^{n \times N}$ is given by the solution of the linear system,

$$\mathbf{L}^{(m)} \cdot \left[\mathbf{D}^{(m)}\right]^{1/2} \cdot \boldsymbol{\Delta}\mathbf{X}^a = \mathbf{W} \in \mathbb{R}^{n \times N}\,, \tag{17}$$

and the columns of $\mathbf{W} \in \mathbb{R}^{n \times N}$ are formed by samples from a multivariate standard normal distribution. Again, since $\mathbf{L}^{(m)}$ is lower triangular, the solution of (17) can be obtained readily.

4 Experimental Results

In this section, we assess the accuracy of the P-EnKF and compare it against that of the LETFK implementation proposed by Hunt in [16]. The numerical model is the Lorenz 96 model [21] which mimics the behaviour of the atmosphere. This model is described by the next set of ordinary differential equations:

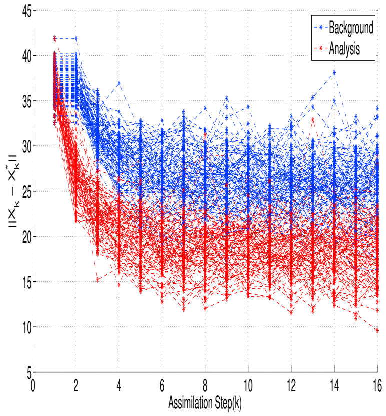
$$\frac{dx_k}{dt} = -x_{k-1} \cdot (x_{k-2} - x_{k+1}) - x_k + F, \text{ for } 1 \leq k \leq n\,, \tag{18}$$

where $n = 40$ is the number of model components and F is an external force. It is well-known that when F equals 8.0, the Lorenz 96 model exhibits a chaotic behaviour which makes it attractive as a toy problem for testing weather prediction methods [22]. The experimental settings are described below:

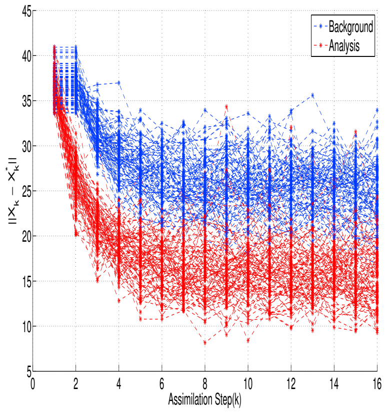
- An initial random solution is propagated for a long time period in order to obtain a reference solution dynamically consistent with the model (18).
- The initial background errors follows a Normal distribution $\mathcal{N}(\mathbf{0},\, \sigma_B \cdot \mathbf{I})$. Three different values for σ_B are considered during the numerical experiments $\sigma_B \in \{0.05,\, 0.10,\, 0.15\}$. This perturbed state is propagated in time in order to make it consistent with the physics and dynamics of the numerical model (18). A similar procedure is performed in order to obtain a background state and an initial ensemble.
- The assimilation windows consists of 15 equidistant observations. The frequency of observations is 0.5 time units which represents 3.5 days in the atmosphere.
- The number of observed components is 50% the dimension of the vector state.
- Three ensemble sizes are tried during the experiments $N \in \{20,\, 40,\, 60\}$.
- As a measure of quality, the L-2 norm of the analysis state and the reference solution is computed across assimilation steps.
- Varying the reference solution, 100 runs are performed for each pair $(N,\, \sigma_B)$.

The average of the error norms of each pair $(N,\, \sigma_B)$ for the LETKF and the P-EnKF implementations are shown in the Table 1. As can be seen, in average across 100 of runs, the performance of the proposed EnKF implementation outperforms that of the LETKF in terms of $L - 2$ norm of the error. Even more, the P-EnKF seems to be invariant to the initial background error σ_B since, in all cases, when the ensemble size is increased a better estimation of the reference state $\mathbf{x}^*$ at different observation times is obtained. This can also obey to the estimation of background error correlations via the modified Cholesky decomposition [19] since it is drsaticaly improved whenever the ensemble size is increased as is pointed out by Bickel and Levina in [23]. In such case, the error decreases by $\mathcal{O}\left(\log(n)/N\right)$. This is crucial in the P-EnKF formulation since estimates of the precision analysis covariance matrix are obtained by rank-one updates on the inverse background error covariance matrix. On the other hand, in the LETKF context, increasing the ensemble size can improve the accuracy of the method but, that is not better than the one shown by the P-EnKF.

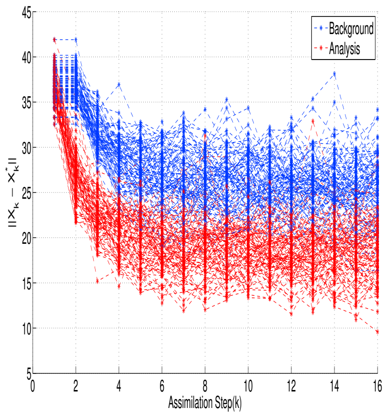
Some plots of the L-2 norm of error for the P-EnKF and the LETKF across different configurations and runs can be seen in figure 1. Note that, the error of the P-EnKF decreases aggressively since the earlier iterations. In the LETKF context, the accuracy is similar to that of the P-EnKF only at the end of the assimilation window.



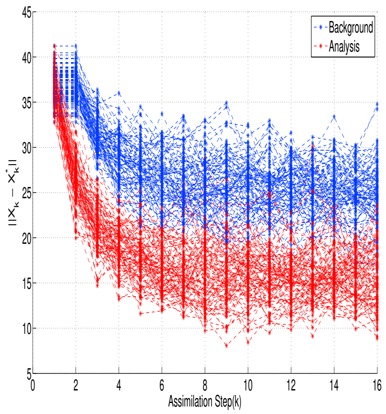
(a) LETKF $\sigma_B = 0.05,\ N = 60$



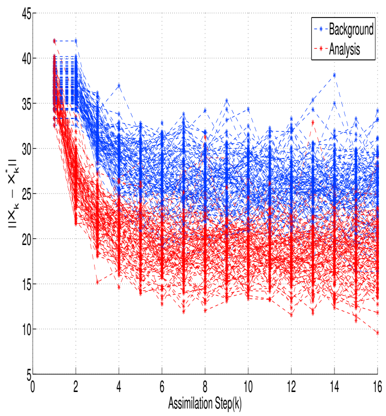
(b) P-EnKF $\sigma_B = 0.05,\ N = 60$



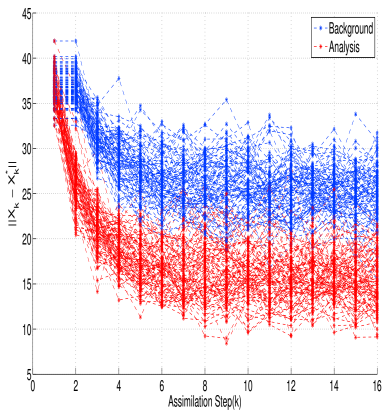
(c) LETKF $\sigma_B = 0.10,\ N = 60$



(d) P-EnKF $\sigma_B = 0.10,\ N = 60$



(e) LETKF $\sigma_B = 0.15,\ N = 60$



(f) P-EnKF $\sigma_B = 0.15,\ N = 60$

Figure 1: $L-2$ norm of the error for the LETKF and the P-EnKF implementations at different observation times. For each configuration, 100 of runs are performed. The assimilation window

σ_B	N	LETKF	P-EnKF
0.05	20	22,6166	21,2591
	40	20,5671	18,2548
	60	20,0567	17,8824
0.10	20	23,1742	21,0725
	40	20,9513	18,3542
	60	18,5048	17,8240
0,15	20	24,8201	20,9059
	40	21,1314	18,1731
	60	20,8487	17,7590

Table 1: Average of $L-2$ norm of errors for 100 of runs of each configuration (σ_B, N) for the compared filter implementations.

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5 Conclusions

We propose a posterior ensemble Kalman filter based on a modified Cholesky decomposition. The proposed method estimates the posterior moments of the error distribution based on an ensemble of model realizations. An estimate of the inverse background error covariance matrix via a modified Cholesky decomposition is updated making use of rank-one matrices with information brought by the data error correlations in order to estimate the precision covariance matrix. This matrix is utilized in order to compute the posterior mode of the error distribution and then, samples are taken about it. This implementation is matrix-free making it attractive for practical implementations. Experimental settings are performed making use of the Lorenz 96 model and different observations and ensemble configurations. The results obtained by the proposed method are compared against those obtained by the local ensemble transform Kalman filter (LETKF). The results reveal that, the use of the proposed implementation can mitigate the impact of sampling errors and even more, the accuracy of the proposed EnKF implementation is similar to that of the LETKF.

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