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**DYNAMICAL AND ALGEBRAIC ANALYSIS OF PLANAR POLYNOMIAL
VECTOR FIELDS LINKED TO ORTHOGONAL POLYNOMIALS****与正交多项式相关的平面多项式矢量场的动力学和代数分析****Jorge Rodríguez Contreras^{a, b, *}, Alberto Reyes Linero^b, Maria Campo Donado^b, Primitivo B. Acosta-Humánez^{c, d}**^a Departamento de Matemática y Estadística, Universidad del Norte
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Attribution License (<http://creativecommons.org/licenses/by/4.0>)***Abstract**

In the present work, our goal is to establish a study of some families of quadratic polynomial vector fields connected to orthogonal polynomials that relate, via two different points of view, the qualitative and the algebraic ones. We extend those results that contain some details related to differential Galois theory as well as the inclusion of Darboux theory of integrability and the qualitative theory of dynamical systems. We conclude this study with the construction of differential Galois groups, the calculation of Darboux first integral, and the construction of the global phase portraits.

Keywords: Darboux First Integral, Differential Galois Theory, Integrability, Orthogonal Polynomial, Polynomials Vector Fields

摘要 在当前的工作中, 我们的目标是建立与正交多项式相关的二次多项式矢量场的一些族的研究, 该正交多项式通过两种不同的观点相互关联, 即定性和代数。我们扩展了这些结果, 这些结果包含与微分加洛瓦理论有关的一些细节, 以及包含达布可积性理论和动力学系统的定性理论。我

们以差分伽罗瓦群的构造，达布克斯第一积分的计算以及整体相像的构造来结束本研究。

关键词: 达布第一积分，微分伽罗瓦理论，可积性，正交多项式，多项式矢量场

I. INTRODUCTION

This paper is a follow-up to [1] and a slight improvement over [2]. To study any process of variation with respect to time, the theory of dynamical systems has been developed, which is also endowed with algebraic and qualitative techniques, among others. Although, in a general case, it is not possible to find the solution of a differential equation that models a specific process, we can identify geometric structures having influence over qualitative properties such as stability and invariant sets attractors, among others, see [3], [4], [5], [6], [7], [8], [9], [10] for further details. In the algebraic sense, E. Picard and E. Vessiot introduced an approach to study linear differential equations based on the Galois theory for polynomials [3], which is known as differential Galois theory or Picard-Vessiot theory [11], [12], [13], [14] for further details. Also, G. Darboux introduced an algebraic theory to analyze the integrability of polynomial vector fields, which is known as Darboux theory of integrability [15]. The final ingredient of this paper corresponds to orthogonal polynomials [16], [17], which are very important in both theoretical and applied mathematics: they contribute to random matrices, approximation theory, trigonometric series, and especially differential equations, among others.

Concerning applications of differential Galois theory to dynamical systems, [18], [19] presented techniques to determine the non-integrability of Hamiltonian systems, which can be found in [1], [18], [19], [20], [21], [22], while [1], [20] presented techniques to study planar polynomial vector fields. In the same way, applications to Quantum Mechanics can be found in [21], [22]. Combinations of algebraic and qualitative techniques to study planar vector fields were presented in [22], [23]. This paper is a sequel of [1], and in particular is an extension of section §4.2. We follow the same structure of papers [22], [23] concerning the algebraic and qualitative techniques to study the polynomial vector fields. We remind the reader that for algebraic analysis, differential Galois theory, and Darboux integrability, we consider vector fields over the complex numbers, while for qualitative analysis we consider the vector fields over the real numbers.

II. PRELIMINARIES

In this section we present the basic theoretical background needed to understand the rest of the paper.

A. Classical Orthogonal Polynomials

The main objects of study in this work are quadratic polynomial differential systems associated to classical orthogonal polynomials. In particular we focus on the sequences of classical orthogonal polynomials of the hypergeometric type—that is, orthogonal polynomials satisfying the differential equation

$$\rho(x)y'' + \tau(x)y' + \lambda y = 0, \quad (1.1)$$

where $\rho(x)$, $\tau(x)$ are polynomials and λ depending on n is given in the next table:

Table 1.
Polynomial list

$\rho(x)$	$\tau(x)$	λ_n
$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$	$n(n + 1 + \alpha + \beta)$
$1 - x^2$	$-2x$	$n(n + 1)$
$1 - x^2$	$-x$	$n2$
$1 - x^2$	$-3x$	$n(n + 2)$
$1 - x^2$	$-(2\alpha + 1)x$	$n(n + 1 + 2\alpha)$
x	$\alpha + 1 - x$	n
x	$1 - x$	n
1	$-2x$	$2n$

Moreover, it is well known that classical orthogonal polynomials can be obtained by Rodrigues formula [16], [17]. In a general form, the constant λ_n can be obtained as follows:

$$\lambda_n = -n \left(\tau' + \frac{n-1}{2} \rho''(x) \right).$$

Thus, the object of study becomes the differential system

$$\begin{aligned} \frac{d\nu}{dt} &= \frac{\lambda_n}{\mu} \rho + (\rho' - \tau)\nu + \mu\nu^2 \\ \frac{dx}{dt} &= \rho \end{aligned}$$

and its associated foliation becomes

$$\frac{d\nu}{dx} = \frac{\lambda_n}{\mu} + \frac{\rho' - \tau}{\rho} \nu + \frac{\mu}{\rho} \nu^2.$$

We claim that $\mu \neq 0$ because we are studying quadratic polynomial vector fields.

B. Critical Points

We recall that a real vector field χ is a function of C^r class where $r \in \mathbb{N} \cup \infty$, ω (if $r = \omega$ we say that the function is analytic). Moreover, $\chi: \Delta \rightarrow \mathbb{R}$ and Δ is an open subset of \mathbb{R} . For instance, the differential system associated to the vector field χ is given by $\dot{x} = \chi(x)$. Now, based on [3], [7], we present the classification of some critical points used in the main results of this paper. The following theorem is concerning hyperbolic critical points.

Theorem 1.1: Let $(0,0)$ be an isolated singular point of the vector field X associated to

$$\begin{aligned}\dot{v} &= av + bx + A(v,x), \\ \dot{x} &= cv + dx + B(v,x),\end{aligned}\quad (1.2)$$

where A and B are analytic in a neighborhood of the origin with $A(0,0) = B(0,0) = DA(0,0) = DB(0,0) = 0$. Let λ_1 and λ_2 be an eigenvalue of the linear part $DX(0,0)$ of the system at the origin. Then the following statements hold:

If λ_1 and λ_2 are real and $\lambda_1\lambda_2 < 0$, then $(0,0)$ is a saddle. If we denote by E_1 and E_2 the eigenspaces of respectively λ_1 and λ_2 , then one can find two invariant analytic curves, tangent respectively to E_1 and E_2 at 0. On one of the points of E_1 the analytic curves are attracted towards the origin, while on one of the points of E_2 the curves are repelled away from the origin. On these invariant curves X is C^∞ -linearizable. There exists a C^∞ coordinate change transforming (1.2) into one of the following normal forms:

$$\begin{aligned}\dot{v} &= \lambda_1 v, \\ \dot{x} &= \lambda_2 x,\end{aligned}$$

in the case $\lambda_1 / \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$, and

$$\begin{aligned}\dot{v} &= v(\lambda_1 + f(v_k, x_k)), \\ \dot{x} &= x(\lambda_2 + g(v_k, x_k)),\end{aligned}$$

in the case $\lambda_1 / \lambda_2 = -k / l \in \mathbb{Q}$ with $k, l \in \mathbb{N}$ and where f, g are function C^∞ . All systems 1.2 are C^0 -conjugate to

$$\begin{aligned}\dot{v} &= v, \\ \dot{x} &= -x.\end{aligned}$$

If λ_1 and λ_2 are real with $|\lambda_2| \geq |\lambda_1|$ and $\lambda_1\lambda_2 > 0$, then $(0,0)$ is a node. If $\lambda_1 > 0$ (respectively < 0), then it is repelling or unstable (respectively attracting or stable). There exists a C^∞ coordinate change transforming (1.2) into

$$\begin{aligned}\dot{x} &= \lambda_1 x, \\ \dot{y} &= \lambda_2 y,\end{aligned}$$

in case $\lambda_1 / \lambda_2 \notin \mathbb{N}$, and into

$$\begin{aligned}\dot{x} &= \lambda_1 x, \\ \dot{y} &= \lambda_2 y + \eta x^m,\end{aligned}$$

for some $\eta = 0$ or 1, in case $\lambda_2 = m\lambda_1$ with $m \in \mathbb{N}$ and $m > 1$. All systems are C^0 -conjugate to

$$\begin{aligned}\dot{x} &= \eta x, \\ \dot{y} &= \eta y,\end{aligned}$$

with $\eta = \pm 1$ and $\lambda_1\eta > 0$.

If $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ with $\alpha, \beta \neq 0$ then $(0,0)$ is a “strong” focus. If $\alpha > 0$ (respectively $\alpha < 0$), it is repelling or unstable (respectively attracting or stable). There exists a C^∞ coordinate change transforming (1.2) into

$$\begin{aligned}\dot{x} &= \alpha x + \beta y, \\ \dot{y} &= -\beta x + \alpha y.\end{aligned}$$

All systems (1.3) are C^0 -conjugate to

$$\begin{aligned}\dot{x} &= \eta x, \\ \dot{y} &= \eta y,\end{aligned}$$

with $\eta = \pm 1$ and $\alpha\eta > 0$.

If $\lambda_1 = \beta i$ and $\lambda_2 = -\beta i$ with $\beta \neq 0$, then $(0,0)$ is a linear center topologically, a weak focus or a center.

The following theorem corresponds to Semi-hyperbolic critical points.

Theorem 1.2: Let $(0,0)$ be an isolated singular point of the vector field X given by

$$\begin{aligned}\dot{x} &= A(x,y) \\ \dot{y} &= \lambda y + B(x,y)\end{aligned}\quad (1.3)$$

where A and B are analytic in a neighborhood of a origin with $A(0,0) = B(0,0) = DA(0,0) = DB(0,0) = 0$ and $\lambda > 0$. Let $y = f(x)$ be the solution of equation $\lambda y + B(x,y) = 0$ in a neighborhood of the point $(0,0)$, and suppose that the function $g(x) = A(x, f(x))$ has the expression $g(x) = a_m x^m + o(x^m)$, where $m \geq 2$ and $a_m \neq 0$. Then, there always exists an invariant analytic curve, called the strong unstable manifold, tangent at 0 to the $\mathbf{0}$ to the y -axis, on which X is analytically conjugate to

$$\frac{dx}{dt} = \lambda x;$$

It represents repelling behavior since $\lambda > 0$. Moreover, the following statements hold.

(i) If m is odd and $a_m < 0$ then $(0,0)$ is a topological saddle. Tangent to the x -axis there is a unique invariant C^∞ curve, called the center manifold, on which X is C^∞ -conjugate to

$$\dot{x} = -x^m(1 + ax^{m-1}),$$

for some $a \in \mathbb{R}$.

If this invariant curve is analytic, then on it X is C^∞ -conjugate to

$$\dot{x} = -x^m(1 + ax^{m-1}), \quad \dot{y} = \lambda y,$$

and is C^0 -conjugate to

$$\dot{x} = -x, \quad \dot{y} = y.$$

(i) if m is odd and $a_m > 0$, the origin is a unstable topological node. Every point not belonging to the strong unstable manifold lies on an invariant C^∞ curve called a center manifold, tangent to the x -axis at the origin, and on which X is a C^∞ -conjugate to

$$\dot{x} = x^m(1 + ax^{m-1}),$$

for some $a \in \mathbb{R}$.

All these center manifolds are mutually infinitely tangent to each other, and hence at most one of them can be analytic, in which case X is C^∞ -conjugate to

$$\dot{x} = x^m(1 + ax^{m-1}),$$

$$\dot{y} = \lambda y$$

And is C^0 -conjugate to

$$\dot{x} = x,$$

$$\dot{y} = y.$$

(ii) If m is even, then $(0,0)$ is a saddle node, that is a singular point whose neighborhood is the union of one parabolic and two hyperbolic sectors. Modulo changing x into $-x$, we suppose that $a_m > 0$. Every point to the right of the strong unstable manifold (side $x > 0$) lies on a invariant C^∞ curve, called a center manifold, tangent to the x -axis at the origin, and on which case X is a C^∞ -conjugate to

$$\dot{x} = x^m(1 + ax^{m-1}),$$

for some $a \in \mathbb{R}$. All these center manifold coincide on the side $x \leq 0$ and are hence infinitely tangent at the origin. At most one of these center manifolds can be analytic, in which case X is C^∞ -conjugate to

$$\dot{x} = x^m(1 + ax^{m-1}),$$

$$\dot{y} = \lambda y,$$

and is C^0 -conjugate to

$$\dot{x} = x^2,$$

$$\dot{y} = \lambda y.$$

The following theorem is concerning to Nilpotent singular points.

Theorem 1.3: Let $(0,0)$ be an isolated singular point of the vector field X given by

$$\dot{x} = x + A(x,y)$$

$$\dot{y} = B(x,y),$$

where A and B are analytic in a neighborhood of the point $(0,0)$ and also $j_1 A(0,0) = j_1 B(0,0) = 0$. Let $y = f(x)$ be the solution of the equations $y + A(x,y) = 0$ in a neighborhood of the point $(0,0)$, and consider $F(x) = B(x, f(x))$ and $G(x) = (\partial A / \partial y + \partial B / \partial x)(x, f(x))$. Then the following holds:

(i) If $F(x) \equiv G(x) \equiv 0$, then the phase portrait of X is given by 1a.

(ii) Si $F(x) \equiv 0$ and $G(x) = bx^n + o(x^n)$ with $n \in \mathbb{N}$, $n \geq 1$ and $b \neq 0$, then the phase portrait of X is given by 1b o c.

(iii) If $G(x) \equiv 0$ and $F(x) = ax^m + o(x^m)$ with $m \in \mathbb{N}$, $m \geq 1$ and $a \neq 0$, then:

If m is odd and $a > 0$, then the origin is a saddle (1d) and if $a < 0$, then it is a center or focus (1e - f).

If m is even the origin of X is a cusp (1h).

(iv) If $F(x) = ax^m + o(x^m)$ and

$G(x) = ax^m + o(x^m)$ with $m, n \in \mathbb{N}$, $m \geq 1$, $n \geq 1$ and $a \neq 0$, $b \neq 0$, then we have:

If m is even, and $m < 2n + 1$, then the origin of is a cusp 1h, or $m > 2n + 1$, then the origin is a saddle-node 1i or j

If m is odd and $a > 0$ then the origin is a saddle 1d.

If m is odd, $a < 0$ and Either $m < 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) < 0$, then the origin is a center or focus (figure 1e, g).

If n is odd and either $m > 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) \geq 0$, then the phase portrait of the origin of X consist of one hyperbolic and one elliptic as in figure (1k).

n is even and either $m > 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) \geq 0$ then the origin of X is a node as in figure 1l, m. The node is attracting if $b < 0$ and repelling if $b > 0$.

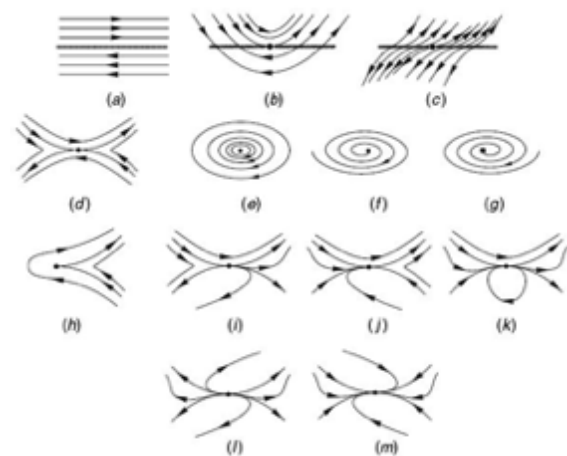


Figure 1. Portraits of phase for 2.8 [3]

For complete study of these theorems see [3].

C. Invariants Curves

Let be the differential polynomial complex system

$$\dot{x} = P(x, y), \quad (1.4)$$

$$\dot{y} = Q(x, y),$$

and $m = \max\{\deg P, \deg Q\}$.

Theorem 1.4: Suppose that a C-polynomial system (1.4) of degree m admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors $K_i = 1, 2, \dots, p$; q exponential factors $\exp(g_i/h_i)$ with cofactors L_j , $j = 1, 2, \dots, q$, and r independent singular points $(x_k, y_k) \in \mathbb{C}^2$ such that $f_i(x_k, y_k) = 0$) then if there exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -s$$

for some $s \in \mathbb{C} \setminus \{0\}$, then the (multivalued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^{st}$$

is an invariant of system (1.4).

For a complete version of this theorem see [13], §8, pp. 219.

The following theorems concern to singular points at infinity, where $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$.

Theorem 1.5: The critical points at infinity for the m th degree polynomial system (1.4) occur at the points $(X, Y, 0)$ over the equator of the Poincaré sphere, being

$$X^2 + Y^2 = 1 \text{ and } XQ_m(X, Y) - YP_m(X, Y) = 0.$$

Theorem 1.6: The flow defined in a neighborhood of any critical point of (1.4) (with mentioned change of variable) over the equator of the Poincaré sphere S^2 , except the points $(0, \pm 1, 0)$, is topologically equivalent to the flow defined by the system:

$$\begin{aligned} \pm \dot{y} &= yz^m P\left(\frac{x}{z}, \frac{y}{z}\right) - z^m Q\left(\frac{x}{z}, \frac{y}{z}\right) \\ \pm \dot{z} &= z^{m+1} P\left(\frac{x}{z}, \frac{y}{z}\right), \end{aligned} \quad (1.5)$$

being the signs determined by the flow on the equator of S^2 such as was determined in Theorem 1.5. Similarly, the flow defined by (1.4) (with the mentioned change of variable) in a neighborhood of any critical point of (1.4) on the equator of S^2 except the points $(\pm 1, 0, 0)$ is topologically equivalent to the flow defined by the system:

$$\begin{aligned} \pm \dot{x} &= xz^m Q\left(\frac{x}{z}, \frac{1}{z}\right) - z^m P\left(\frac{x}{z}, \frac{1}{z}\right) \\ \pm \dot{z} &= z^{m+1} P\left(\frac{x}{z}, \frac{1}{z}\right), \end{aligned} \quad (1.6)$$

the signs being determined by the flow on the equator of S^2 as determined in the theorem (1.5). This theory can be study in detail on [3], [7].

III. RESULTS AND DISCUSSION

In this section we demonstrate the main results of the paper. We begin by presenting some results of orthogonal polynomials theory from a Galoisian point of view. The following proposition relates the classical Galois theory with orthogonal polynomials.

Proposition 2.1: If P_n is an orthogonal polynomial, then for the splitting field of the polynomial $P_n(x)$ over \mathbb{R} , $(\mathbb{R}\{P_n(x)\})$; we know that $\mathbb{R}\{P_n(x)\} = \mathbb{R}$.

Proof: As the roots $\alpha_1, \dots, \alpha_n$ of any orthogonal polynomial P_n of degree n are real and distinct, then

$$\mathbb{R}\{P_n\} = \mathbb{R}[\alpha_1, \dots, \alpha_n].$$

Taking the integral domain $\mathbb{R}[\alpha_1]$. By definition, we know that

$$\mathbb{R}[\alpha_1] = \{f(\alpha_1)/f(x) \in \mathbb{R}[x]\}.$$

Thus, $f(\alpha_1) \in \mathbb{R}$. In this way $\mathbb{R}[\alpha_1, \dots, \alpha_n] = \mathbb{R}$.

Remark 2.1: From the previous proposition, we can observe that if we take the real members as the base field, then the splitting field of any orthogonal polynomial is again the real numbers. That is, the extension $L = \mathbb{R}\{P_n\} = \mathbb{R}$, and therefore the Galois group of the polynomial is $G(L \setminus \mathbb{R}) = \{f: f(x) = x, \forall x \in \mathbb{R}\} = Id$.

The following proposition appears in [5], §4.2, and it is included jointly with the proof for completeness.

Proposition 2.2: If we consider two polynomials $\rho(x)$, $\tau(x)$ and the parameter λ_n from the previous table, then for any μ , the Riccati type differential equation

$$\frac{dv}{dx} = \frac{\lambda_n}{\mu} + \frac{\rho' - \tau}{\rho} v + \frac{\mu}{\rho} v^2, \quad (2.1)$$

can be transformed into the hypergeometric type equation (1.1).

$$\rho(x)y'' + K_1 P_1 y' + \lambda_n y = 0$$

Proof: Making the change of variable $w = \mu v$, we obtain

$$\begin{aligned} \frac{dw}{dx} &= \mu \frac{dv}{dx} \\ &= \lambda_n + \frac{\rho' - \tau}{\rho} \mu v + \frac{1}{\rho} \mu^2 v^2 \\ &= \lambda_n + \frac{\rho' - \tau}{\rho} w + \frac{1}{\rho} w^2, \end{aligned}$$

obtaining the differential equation

$$\frac{dw}{dx} = \lambda_n + \frac{\rho' - \tau}{\rho} w + \frac{1}{\rho} w^2$$

Now if we take

$$w = -\rho \frac{y'}{y}$$

then

$$y \frac{dw}{dx} + w \frac{dy}{dx} = -\rho' y' - \rho y'' \quad (2.2)$$

On the other hand,

$$\begin{aligned} y \frac{dw}{dx} + w \frac{dy}{dx} &= y \left[\lambda_n - \frac{\rho' - \tau}{\rho} \left(\rho \frac{y'}{y} \right) + \frac{1}{\rho} \left(\rho \frac{y'}{y} \right)^2 \right] + \left(\rho \frac{y'}{y} \right) y' \\ &= y \lambda_n - \rho' y' + \tau y'. \end{aligned}$$

This is,

$$y \frac{dw}{dx} + w \frac{dy}{dx} = y \lambda_n - \rho' y' + \tau y' \quad (2.3)$$

Now by (2.2) and (2.3), we have

$$\begin{aligned} y \lambda_n - \rho' y' + \tau y' &= -\rho' y' - \rho y'', \\ \rho y'' + \tau y' + \lambda_n y &= 0. \end{aligned}$$

In this way we can associate a polynomial system in the plane to each family of classical orthogonal polynomials in table 2.2

The following theorem appears in [20], §4.2, and it is included jointly with the proof for completeness.

Theorem 2.3: Let $\rho(x)$, $\tau(x)$ and λ_n as in the previous proposition. For any $\mu \neq 0$, the quadratic polynomial vector field corresponding to the system

$$\begin{aligned} \frac{dv}{dt} &= \frac{\lambda_n}{\mu} \rho + (\rho' - K_1 P_1) v + \mu v^2, \\ \frac{dx}{dt} &= \rho \end{aligned} \quad (2.4)$$

has an invariant algebraic curve of the form $\mu v P_n(x) + \rho(x) P'_n(x) = 0$, where P_n is any classical orthogonal polynomial associated to $\rho(x)$, $\tau(x)$ and λ_n .

Proof: The differential equation associated with the polynomial system (2.4) is:

$$\frac{dv}{dx} = \frac{\lambda_n}{\mu} + \frac{\rho' - \tau}{\rho} v + \frac{\mu}{\rho} v^2$$

which, by Proposition 2.2 can be transformed in the hypergeometric equation (1.1); and for each $n \in \mathbb{Z}^+$, we have the solution $y_n = P_n$, which is a classical orthogonal polynomial associated with functions $\rho(x)$, $\tau(x)$ and the parameter λ_n .

$$\rho(x) P_n'' + \tau P_n' + \lambda_n P_n = 0 \quad (2.5)$$

Table 2.

Family of classical orthogonal polynomials

Family	v^*	x^*
(α, β)	$\frac{\lambda_n}{\mu} (1 - x^2) + (\alpha - \beta + (\alpha - \beta)x)v + \mu v^2$	$1 - x^2$
P_n	$\frac{\lambda_n}{\mu} (1 - x^2) + \mu v^2$	$1 - x^2$
T_n	$\frac{\lambda_n}{\mu} (1 - x^2) - xv + \mu v^2$	$1 - x^2$
U_n	$\frac{\lambda_n}{\mu} (1 - x^2) + xv + \mu v^2$	$1 - x^2$
$C_n^{(\alpha)}$	$\frac{\lambda_n}{\mu} (1 - x^2) + (2\alpha - 1)xv + \mu v^2$	$1 - x^2$
$L_n^{(\alpha)}$	$\frac{\lambda_n}{\mu} x + (-\alpha + x)v + \mu v^2$	x
L_n	$\frac{\lambda_n}{\mu} x + xv + \mu v^2$	x
H_n	$\frac{\lambda_n}{\mu} + 2xv + \mu v^2$	1

Let X be the vector field associated with the differential system (2.4). Now, for n fixed, we consider the polynomial $f(v, x) = \mu v P_n(x) + \rho P'_n(x)$, and we show that it is irreducible and satisfies $Xf = Kf$, where K is the cofactor of the invariant curve $f = 0$.

We know that both $P_n(x)$ and $P'_n(x)$ do not have common factors because the roots of the orthogonal polynomials are simple. In addition, with $\rho(x)$ defined for each family of classical orthogonal polynomials, we have that both $\rho(x)$ and $P_n(x)$ do not share roots, because the roots of orthogonal polynomials remain within the range (a, b) . In fact:

♣ in the Jacobi polynomial, $\rho(x) = 1 - x^2$ whose roots are not in the interval $(-1, 1)$;

♣ in the Laguerre polynomials, $\rho(x) = x$ whose root is not in the interval $(0, \infty)$; and

♣ in the Hermite polynomials, $\rho(x) = 1$; hence, the polynomial $f(v, x) = \mu v P_n(x) + \rho P'_n(x)$ is irreducible.

On the other hand, using the differential field associated with the differential system and (2.5), we have that

$$\begin{aligned} Xf &= \left(\frac{\lambda_n}{\mu} \rho + (\rho' - K_1 P_1) v + \mu v^2 \right) \frac{\partial f}{\partial v} + \rho \frac{\partial f}{\partial x} \\ &= \left(\frac{\lambda_n}{\mu} \rho + (\rho' - K_1 P_1) v + \mu v^2 \right) \mu P_n + \rho (\mu v P'_n + \rho' P'_n + \rho^2 P''_n) \\ &= \rho' (\mu v P_n + \rho P'_n) + \mu v (\mu v P_n + \rho P'_n) - \mu K_1 P_1 v P_n + \rho (\rho P''_n + \lambda_n P_n) \\ &= \rho' (\mu v P_n + \rho P'_n) + \mu v (\mu v P_n + \rho P'_n) - \mu K_1 P_1 v P_n - K_1 P_1 \rho P'_n \\ &= \rho' (\mu v P_n + \rho P'_n) + \mu v (\mu v P_n + \rho P'_n) - K_1 P_1 (\mu v P_n + \rho P'_n) \\ &= (\rho' + \mu v - K_1 P_1) (\mu v P_n + \rho P'_n) \\ Xf &= (\rho' + \mu v - K_1 P_1) f \end{aligned}$$

The above implies that $\mu v P_n(x) + \rho P_n'(x) = 0$ is an invariant curve for the system (2.4).

The following proposition is entirely a contribution of this paper:

Proposition 2.4: The quadratic polynomial system (2.6)

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}(1-x^2) + avx + bv + \mu v^2 = P(v, x) \\ \dot{x} = 1 - x^2 \end{cases}$$

has an invariant of Darboux in the form

$$I(v, x, t) = \frac{\sqrt{x-1}}{\sqrt{x+1}} e^t.$$

Proof: The algebraic curves

$$f_1(v, x) = x + 1 = 0, f_2(v, x) = x - 1 = 0$$

are invariant algebraic curves of the system (2.6) with cofactors

$$K_1(v, x) = 1 - x, K_2(v, x) = -1 - x,$$

respectively.

In fact, since, for this system, the vector field is defined as

$$X = P(v, x) \frac{\partial}{\partial v} + (1 - x^2) \frac{\partial}{\partial x},$$

we obtain

$$X(f_1) = (1 - x)f_1 \text{ and } X(f_2) = (-1 - x)f_2.$$

Now using theorem 1.4, taking $s = 1$,

$$\lambda_1 K_1 + \lambda_2 K_2 = -1,$$

we obtain

$$\lambda_1 = -1/2, \lambda_2 = 1/2.$$

Thus, we obtain the Darboux invariant

$$I(v, x, t) = \frac{\sqrt{x-1}}{\sqrt{x+1}} e^t.$$

Now we will study the phase portraits on the Poincaré disk of the polynomial systems associated with the classical orthogonal polynomials, which is one of the main contributions of this paper.

Proposition 2.5: The phase portrait on the Poincaré disk of any quadratic polynomial system

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}(1-x^2) + avx + \mu v^2 \\ \dot{x} = 1 - x^2 \end{cases} \quad (2.7)$$

with $\mu \neq 0$, $\lambda_n > 0$ and $a \in \mathbb{R}$ is topologically equivalent to some of the phase portraits described in Figure 2.

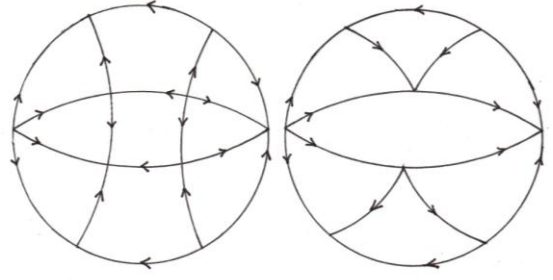


Figure 2. Phase portraits for the system 2.6

Proof: In the finite plane, the singular points of the system are $(0, 1), (0, -1), (-a/\mu, 1), (a/\mu, -1)$.

Two cases are possible: If $a \neq 0$, there are four singular points, and if $a = 0$, there are only two singular points.

Case 1: $a \neq 0$

In the finite plane, there are four singular points:

$$DX(v, x) = \begin{bmatrix} ax + 2\mu v & -2\frac{\lambda_n}{\mu}x + av \\ 0 & -2x \end{bmatrix}$$

By evaluating this matrix in each of the singular points, we obtain

$$DX(0, 1) = \begin{bmatrix} a & -2\lambda_n/\mu \\ 0 & -2 \end{bmatrix}$$

$$DX(-a/\mu, 1) = \begin{bmatrix} -a & -(2\lambda_n + a^2)/\mu \\ 0 & -2 \end{bmatrix}$$

$$DX(0, -1) = \begin{bmatrix} -a & 2\lambda_n/\mu \\ 0 & 2 \end{bmatrix}$$

$$DX(a/\mu, -1) = \begin{bmatrix} a & (2\lambda_n - a^2)/\mu \\ 0 & 2 \end{bmatrix}$$

Therefore, there are two saddle points and two nodes in the finite plane; one of each is stable, and the other is unstable.

Case 2: $a = 0$

There are only two singular points in the finite plane. The Jacobian matrix of the system (2.7), with $a = 0$, is

$$DX(v, x) = \begin{bmatrix} 2\mu v & -2\frac{\lambda_n}{\mu}x \\ 0 & -2x \end{bmatrix}$$

$$DX(0, 1) = \begin{bmatrix} 0 & -2\lambda_n/\mu \\ 0 & -2 \end{bmatrix}$$

$$DX(0, -1) = \begin{bmatrix} 0 & 2\lambda_n/\mu \\ 0 & 2 \end{bmatrix}$$

That is, the singular points $(0,1)$ and $(0,-1)$ are semi-hyperbolic.

Using the theorem (1.3), we are able to analyze the behavior of previous singular points in a neighborhood of the origin. We must translate these points to the origin of the coordinated plane and, after transforming the system, rewrite it in a normal way (using the normal forms theorem).

When we perform the following translation, the result will be a system topologically equivalent to (2.7):

$$\text{since } \tilde{x} = x - 1, v = v,$$

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}(-2\tilde{x} - \tilde{x}^2) + \mu v^2 \\ \dot{\tilde{x}} = -2\tilde{x} - \tilde{x}^2, \end{cases}$$

then,

$$\tilde{v} = v - \frac{\lambda_n}{\mu}\tilde{x},$$

$$\begin{cases} \dot{\tilde{v}} = \frac{\lambda_n^2}{\mu}\tilde{x}^2 + 2\lambda\tilde{v}\tilde{x} + \mu\tilde{v}^2 \\ \dot{\tilde{x}} = -2\tilde{x} - \tilde{x}^2. \end{cases}$$

This last system is topologically equivalent to the system (2.7) and meets the hypothesis of the theorem for semi-hyperbolic points. If we take

$$A(\tilde{v}, \tilde{x}) = \frac{\lambda_n^2}{\mu}\tilde{x}^2 + 2\lambda\tilde{v}\tilde{x} + \mu\tilde{v}^2$$

and

$$B(\tilde{v}, \tilde{x}) = -\tilde{x}^2,$$

then

$$\tilde{x} = f(\tilde{v}) = -\frac{1}{2}\tilde{v}^2 + o(\tilde{v}^2)$$

is the solution of

$$-2\tilde{x} + B(\tilde{v}, \tilde{x}) = 0$$

near of origin.

$$\text{Now, } g(\tilde{v}) = A(v, \tilde{f}(\tilde{v})) = \mu\tilde{v}^2 + o(\tilde{v}^2)$$

because the lowest-order term of the function $g(\tilde{v})$ is even, the singular point $(0,1)$ is a saddle-node point.

Now, for the semi-hyperbolic point $(0,-1)$ we make transformations

$$\begin{aligned} v &= v, & \tilde{x} &= x + 1 & \text{and} \\ \tilde{x} &= \tilde{x} & \tilde{v} &= v - \frac{\lambda_n}{\mu}\tilde{x}, \end{aligned}$$

obtaining that $(0,-1)$ is a saddle-node point.

Now, we will analyze the singular points in infinity using the transformations on the Poincaré sphere [6].

The flow, defined by study system 2.7, on the equator of the Poincaré sphere, excluding

$(\pm 1, 0, 0)$, is topologically equivalent to the flow defined by the system

$$\begin{cases} \dot{v} = -\lambda_n/\mu + (a+1)v + \mu v^2 + \lambda_n/\mu z^2 - v z^2 \\ \dot{x} = -z^3 + z \end{cases},$$

whose singular points to study are:

$$(v_1, 0) = \left(\frac{-(a+1) + \sqrt{(a+1)^2 + 4\lambda_n}}{2\mu}, 0 \right)$$

$$(v_2, 0) = \left(\frac{-(a+1) - \sqrt{(a+1)^2 + 4\lambda_n}}{2\mu}, 0 \right)$$

and

$$DX(v, z) = \begin{bmatrix} (a+1) + 2\mu - z^2 - v & 2\frac{\lambda_n}{\mu}z - 2vz \\ 0 & -3z^2 + 1 \end{bmatrix},$$

then,

$$DX(v_1, 0) = \begin{bmatrix} \sqrt{(a+1)^2 + 4\lambda_n} & 0 \\ 0 & 1 \end{bmatrix} \text{ and}$$

$$DX(v_2, 0) = \begin{bmatrix} -\sqrt{(a+1)^2 + 4\lambda_n} & 0 \\ 0 & 1 \end{bmatrix},$$

which indicates that, $(v_1, 0)$ is an unstable node and $(v_2, 0)$ is a saddle point.

The flow defined by the study system on the equator of the Poincaré sphere, excluding $(0, \pm 1, 0)$, is topologically equivalent to the flow defined by the system

$$\begin{cases} \dot{x} = -\frac{\lambda_n}{\mu}xz^2 + \frac{\lambda_n}{\mu}x^3 - \mu x + z^2 - (a+1)x^2 \\ \dot{z} = -\frac{\lambda_n}{\mu}z^3 + \frac{\lambda_n}{\mu}x^2z - axz - \mu z \end{cases}$$

in which it is only necessary to study the behavior of the singular point, the origin:

$$DX(x, z) = \begin{bmatrix} -\frac{\lambda_n}{\mu}z^2 + 3\frac{\lambda_n}{\mu}x^2 - \mu - 2(a+1)x & -2\frac{\lambda_n}{\mu}xz + 2z \\ 2\frac{\lambda_n}{\mu}xz - az & -3\frac{\lambda_n}{\mu}z^2 + \frac{\lambda_n}{\mu}x^2 - ax - \mu \end{bmatrix}$$

then, evaluating the Jacobian matrix in the point $(0,0)$, we get:

$$DX(0, 0) = \begin{bmatrix} -\mu & 0 \\ 0 & -\mu \end{bmatrix},$$

which means the origin of this last system is a node, and its stability depends on the sign of μ .

Remark 2.2: For specific values of parameter a , phase portraits are obtained for the polynomial systems associated with the following orthogonal polynomials:

$$a = 0, P_n a = -1, T_n a = 1, U_n,$$

$$a = 2\alpha - 1, \text{ and } C_n(\alpha).$$

Proposition 2.6: The phase portrait on the Poincaré disk of any quadratic polynomial system is as follows:

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}x + av + bv x + \mu v^2 \\ \dot{x} = x \end{cases}, \quad (2.8)$$

where $\mu \neq 0$, $\lambda_n > 0$ and $a, b \in \mathbb{R}$ are topologically equivalent to some of the phase portraits described in Figure 3.

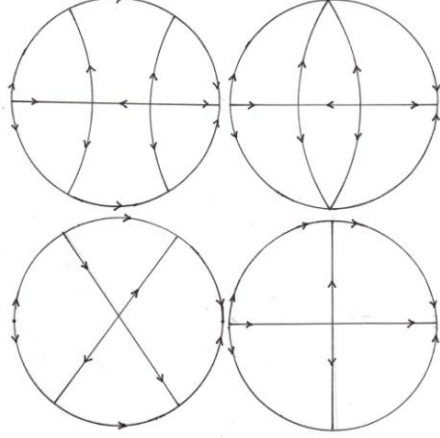


Figure 3. Portraits of phase for 2.8

Proof: In this system the singular points in the finite plane have the form $(0,0)$ and $(-\frac{a}{\mu}, 0)$. That is, if $a = 0$ there is only one singular point and if $a \neq 0$, there are two singular points.

The Jacobian Matrix of the system is

$$DX(v, x) = \begin{bmatrix} a + bx + 2\mu v & \frac{\lambda_n}{\mu}x + bv \\ 0 & 1 \end{bmatrix}$$

Case 1: Laguerre associate $a \neq 0$

$$DX(0,0) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \quad DX(-a/\mu, 0) = \begin{bmatrix} -a & -ab/\mu \\ 0 & 1 \end{bmatrix}$$

Indistinct of the sign of a , in the finite plane, there is a saddle point and an unstable node.

Case 2: Laguerre $a = 0$

$$DX(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This implies that the origin is a singular semi-hyperbolic point. Making the transformations

$$\tilde{v} = v - \frac{\lambda_n}{\mu}x, \quad x = x$$

we get the following system, which is, topologically equivalent to (2.8):

$$\begin{cases} \dot{\tilde{v}} = \frac{\lambda_n(b + \lambda_n)}{\mu}x^2 + (b + 2\lambda_n)\tilde{v}x + \mu v^2 \\ \dot{x} = x \end{cases}$$

Applying the theorem for semi-hyperbolic points, we use

$$A(\tilde{v}, x) = \frac{\lambda_n(b + \lambda_n)}{\mu}x^2 + (b + 2\lambda_n)\tilde{v}x + \mu v^2$$

and

$$B(\tilde{v}, 0) = 0.$$

Then $x = f(\tilde{v}) = 0$ is the solution of equation $x + B(\tilde{v}, 0) = 0$, in a neighborhood of origin.

Now,

$$g(\tilde{v}) = A(\tilde{v}, 0) = \mu \tilde{v}^2 + o(\tilde{v}^2);$$

therefore, the origin is a saddle-node.

Again, the singular points in infinity will be analyzed using the transformations on the Poincaré sphere.

The flow defined by study system 2.8 on the equator of the Poincaré sphere, excluding $(\pm 1, 0, 0)$, is topologically equivalent to the flow defined by the system:

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}z + bv + (a-1)vz + \mu v^2 \\ \dot{z} = -z^2 \end{cases},$$

whose singular points are: $(0,0)$ and $(-b/\mu, 0)$. If $b \neq 0$ there are two singular points. If $b = 0$ there is only one singular point.

The Jacobian matrix associated with this last system is

$$DX(v, z) = \begin{bmatrix} b + (a-1)z + 2v\mu & \\ & 0 \\ \frac{\lambda_n}{\mu} + (a-1)v & \\ & -2z \end{bmatrix} \quad (2.9)$$

Case 1: Laguerre and Laguerre associate $b \neq 0$

$$DX(0,0) = \begin{bmatrix} b & \frac{\lambda_n}{\mu} \\ 0 & 0 \end{bmatrix}$$

$$DX(-b/\mu, 0) = \begin{bmatrix} -b & \frac{\lambda_n + b(1-a)}{\mu} \\ 0 & 0 \end{bmatrix}.$$

That is, $(0,0)$ and $(-b/\mu, 0)$ are semi-hyperbolic points.

To express the system (2.9) in canonical form, and thus be able to apply the theorem for semi-hyperbolic points, we perform the following transformations:

$$\tilde{v} = \frac{\lambda_n}{\mu}z + bv \quad z = z,$$

obtaining the following system, which is topologically equivalent to (2.9):

$$\begin{cases} \dot{\tilde{v}} = b\tilde{v} + \frac{\lambda_n(-a + \lambda_n/b)}{\mu}z^2 + (a-1-2\lambda_n/b)\tilde{v}z + \frac{\mu}{b}\tilde{v}^2 \\ \dot{z} = -z^2 \end{cases},$$

where

$$B(\tilde{v}, z) = \frac{\lambda_n(-a + \lambda_n/b)}{\mu}z^2 + (a-1-2\lambda_n/b)\tilde{v}z + \frac{\mu}{b}\tilde{v}^2$$

$$A(\tilde{v}, z) = -z^2.$$

Let $\tilde{v} = f(z)$ the solution of equation and $b\tilde{v} + B(\tilde{v}, z) = 0$ in a neighborhood of origin.

Then,

$$g(z) = A(f(z), z) = -z^2,$$

so (0,0) is a saddle-node.

For the point $(-b/\mu, 0)$, we will successively use the following transformations:

$$\begin{aligned} \tilde{v} &= v + \frac{b}{\mu} & \tilde{v} &= v + \frac{b}{\mu} & z &= z, \text{ and} \\ \tilde{v} &= \frac{\lambda_n + b(1-a)}{\mu}z - b\tilde{v} & z &= z \end{aligned}$$

obtaining the system topologically equivalent to (2.9):

$$\dot{v}^- = -bv^- + B^-(v^-, z)$$

$$\dot{z} = -z^2$$

where

$$B(0,0) = DB(0,0) = 0$$

and

$$A(v^-, z) = -z^2.$$

Let $v^- = f(z)$ the solution of the equation $-bv^- + B(v^-, z) = 0$ in a neighborhood of the origin of this latter system. Then

$$g(z) = A(f(z), z) = -z^2.$$

Therefore, the point $(-b/\mu, 0)$ is a saddle-node.

Case 2: $b = 0$

$$DX(0,0) = \begin{bmatrix} 0 & \frac{\lambda_n}{\mu} \\ 0 & 0 \end{bmatrix}.$$

That is, the origin is a unique nilpotent point for this system. We make the transformation

$$\tilde{v} = \frac{\mu}{\lambda_n}v, \quad z = z,$$

obtaining the system topologically equivalent to the system (2.9):

$$\begin{cases} \dot{\tilde{v}} = z + (a-1)\tilde{v}z + \lambda_n\tilde{v}^2 \\ \dot{z} = -z^2 \end{cases}.$$

This last system fulfills the conditions of a theorem for singular nilpotent points where

$$A(\tilde{v}, z) = (a-1)\tilde{v}z + \lambda_n\tilde{v}^2 \text{ and } B(\tilde{v}, z) = -z^2.$$

Otherwise, $z = f(\tilde{v}) = (1 - \lambda_n - a)\tilde{v}^2 + o(\tilde{v}^2)$ is the solution to equation

$$z + A(\tilde{v}, z) = 0$$

in a neighborhood of the origin.

Then,

$$F(\tilde{v}) = B(\tilde{v}, f(\tilde{v})) = -(1 - \lambda_n - a)^2\tilde{v}^4 + o(\tilde{v}^4)$$

$$G(\tilde{v}) = \left(\frac{\partial A}{\partial \tilde{v}} + \frac{\partial B}{\partial z} \right) (\tilde{v}, f(\tilde{v})) = 2\lambda_n\tilde{v} + o(\tilde{v})$$

In this case $m = 4$ y $n = 1$. Since m is even and $m > 2n + 1$, the origin is a saddle-node.

For the infinity, the flow defined by the system on the equator Poincaré sphere, excluding $(0, \pm 1, 0)$, is topologically equivalent to the flow defined by the system

$$\begin{cases} \dot{x} = (1-a)xz - \frac{\lambda_n}{\mu}x^2z - bx^2 - \mu x \\ \dot{z} = -\frac{\lambda_n}{\mu}xz^2 - az^2 - xz - \mu z, \end{cases}$$

in which it is only necessary to study the behavior of the singular point, the origin:

$$DX(x, z) =$$

$$\begin{bmatrix} (1-a)z - 2\frac{\lambda_n}{\mu}xz - 2bx - \mu & (1-a)x - \frac{\lambda_n}{\mu}x^2 \\ -\frac{\lambda_n}{\mu}z^2 - z & -2\frac{\lambda_n}{\mu}xz - 2az - x - \mu \end{bmatrix}.$$

In (0,0),

$$DX(0,0) = \begin{bmatrix} -\mu & 0 \\ 0 & -\mu \end{bmatrix},$$

that is, the origin of this last system is a node, and its stability depends on the sign of μ .

Remark 2.3: In the previous proposition, for specific values of parameters a and b , the phase portraits for the polynomial systems associated with the following orthogonal polynomials are obtained:

$$\begin{aligned} a &= 0, & b &= 1 & L_n \\ a &= -\alpha, & b &= 1 & L_n^{(\alpha)}. \end{aligned}$$

To finish this section, we compute the differential Galois group and the elements of Darboux integrability to the quadratic polynomial vector field related with the Chebyshev differential equation.

Proposition 2.7: For the Chebyshev differential equation

$$y'' - \frac{x}{1-x^2}y' + \frac{\lambda_n}{1-x^2}y = 0, \quad (2.10)$$

where $\lambda_n = n^2$, $n \in \mathbb{N}$, and the following statements are true:

(1) $G(L/K)$ of the Chebyshev equation is isomorphic to \mathbb{Z}_2 , where $K = \mathbb{C}(x)$.

(2) The first integrals of fields

$$\begin{cases} \dot{w} = -2 - 4\lambda_n + (4\lambda_n - 1)x^2 - 4(1-x^2)^2w^2 \\ \dot{x} = 4(1-x^2)^2 \end{cases} \quad (2.11)$$

and

$$\begin{cases} \dot{v} = \frac{\lambda_n}{\mu}(1-x^2) - xv + \mu v^2 \\ \dot{x} = 1-x^2 \end{cases}$$

associated with the Chebyshev equation, are:

$$I(w, z) = \frac{-w + \frac{U'_{n-1}}{U_{n-1}} - \frac{3x}{2(1-x^2)}}{-w + \frac{T'_n}{T_n} - \frac{x}{2(1-x^2)}} \cdot \frac{U_{n-1}}{T_n} \sqrt{1-x^2}$$

and

$$I(v, z) = \frac{U'_{n-1}(1-x^2) + U_{n-1}(\mu v - x)}{T'_n(1-x^2) + \mu T_n v x} \cdot \sqrt{1-x^2}$$

Proof: (1) It is known that $y_1 = T_n$ and $y_2 = U_{n-1} \sqrt{1-x^2}$ are two linearly independent solutions for equation (2.10). If we take the differential body $K = \mathbb{C}(x)$ of all the rational functions of variable x , we consider the extension of the field $L = K[1-x^2]$. To calculate the differential Galois group of equation (2.10), all differential automorphisms in the extension must be calculated for L . That is, find a matrix

$$A_\phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\phi \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A_\phi \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

By matrix operations we have:

$$\phi(y_1) = ay_1 + by_2, \quad \phi(y_2) = cy_1 + dy_2$$

On the other hand, $y_1, y_2 \in \mathbb{C}(x)$ and ϕ are automorphisms, then we get

$$\phi(y_1) = y_1, \quad \phi(y_2) = cy_2$$

when $c^2 = 1$. Then we can conclude that

$$A_\phi = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

This is,

$$D(L/K) \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \cong \mathbb{Z}_2$$

$$b_1(x) = \frac{-x}{1-x^2} \text{ y } b_0(x) = \frac{\lambda_n}{1-x^2}$$

(2) If in the equation (2.10) we consider, then transformation

$$y = ze^{-\frac{1}{2} \int b_1(x) dx}$$

allows us to obtain the reduced second-order equation

$$z'' = \left(\frac{-2 - x^2 - 4\lambda_n(1-x^2)}{4(1-x^2)^2} \right) z, \quad (2.13)$$

With

$$z = y(1-x^2)^{\frac{1}{4}}$$

Since $y_1 = T_n$ and $y_2 = U_{n-1}$ are linearly independent solutions of the Chebyshev equation, then:

$$z_1 = T_n(1-x^2)^{\frac{1}{4}}, \quad z_2 = U_{n-1}(1-x^2)^{\frac{3}{4}},$$

are linearly independent solutions of the reduced second-order equation (2.13).

On the other hand, the differential equation associated with the system (2.11) has the form:

$$w' = \frac{-2 - 4\lambda_n + (-1 + 4\lambda_n)x^2}{4(1-x^2)^2} - w^2$$

and applying the transformation $w = \frac{z'}{z}$, is equivalent to the equation (2.13). From this the solutions of this last equation are given by:

$$w_1 = \frac{z'_1}{z_1} = (\ln z_1)' = \frac{T'_n}{T_n} - \frac{x}{2(1-x^2)}$$

and

$$w_2 = \frac{z'_2}{z_2} = (\ln z_2)' = \frac{U'_{n-1}}{U_{n-1}} - \frac{3x}{2(1-x^2)}.$$

Then, by Lemma 1 of [6], we get that the first integral of the system (2.11) has the form:

$$I(w, x) = \frac{-w(x) + w_2(x)}{-w(x) + w_1(x)} \cdot e^{\left(\int (w_2(x) - w_1(x)) dx \right)}.$$

This is,

$$I(w, x) = \frac{-w + \frac{U'_{n-1}}{U_{n-1}} - \frac{3x}{2(1-x^2)}}{-w + \frac{T'_n}{T_n} - \frac{x}{2(1-x^2)}} \cdot \frac{U_{n-1}}{T_n} \sqrt{1-x^2}$$

Now to find the first integral of the system (2.12), it can be noticed that the foliation of this system and the foliation of the system (2.11) are

$$v' = \frac{\lambda_n}{\mu} - \frac{x}{1-x^2}v + \frac{\mu}{1-x^2}v^2,$$

$$w' = \frac{-2 - 4\lambda_n + (4\lambda_n - 1)x^2}{4(1 - x^2)^2} - w^2.$$

which are related through the transformation

$$w = -\frac{x}{2(1 - x^2)} - \frac{\mu v}{1 - x^2}.$$

Therefore, replacing, we obtain

$$I(v, x) = \frac{\frac{x}{2(1-x^2)} + \frac{\mu v}{1-x^2} + \frac{U'_{n-1}}{U_{n-1}} - \frac{3x}{2(1-x^2)}}{\frac{x}{2(1-x^2)} + \frac{\mu v}{1-x^2} + \frac{T'_n}{T_n} - \frac{x}{2(1-x^2)}} \cdot \frac{U_{n-1}}{T_n} \sqrt{1-x^2}$$

and after simplifying we get the first integral described for the system (2.12).

IV. CONCLUSIONS

In this paper, we studied algebraically through differential Galois theory and Darboux theory of integrability, as well as qualitatively through the analysis of critical points, some quadratic polynomial vector fields related with classical orthogonal polynomials.

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