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Research Article

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Algebraic and qualitative remarks about the family $yy' = (\alpha x^{m+k-1} + \beta x^{m-k-1})y + \gamma x^{2m-2k-1}$

Abstract: The aim of this paper is the analysis, from algebraic point of view and singularities studies, of the 5-parametric family of differential equations

$$yy' = (\alpha x^{m+k-1} + \beta x^{m-k-1})y + \gamma x^{2m-2k-1}, \quad y' = \frac{dy}{dx}$$

where $a, b, c \in \mathbb{C}$, $m, k \in \mathbb{Z}$ and

$$\alpha = a(2m + k) \quad \beta = b(2m - k), \quad \gamma = -(a^2 m x^{4k} + c x^{2k} + b^2 m).$$

This family is very important because include Van Der Pol equation. Moreover, this family seems to appear as exercise in the celebrated book of Polyanin and Zaitsev. Unfortunately, the exercise presented a typo which does not allow to solve correctly it. We present the corrected exercise, which corresponds to the title of this paper. We solve the exercise and afterwards we make algebraic and of singularities studies to this family of differential equations.

Keywords: critical points, integrability, Gegenbauer equation, Legendre equation, Liénard equation

MSC: Primary 12H05; Secondary 34C99

Introduction

Dynamical systems is a topic of interest for a large number of theoretical physicist and mathematicians due to the seminal works of H. Poincaré. It is well known that any dynamical system is a system which evolves in the time. H. Poincaré introduced the qualitative approach to study dynamical systems, which has been useful to study theoretical aspects and applications to biology, chemistry, physics, among others, see [1–7].

On another hand, E. Picard and E. Vessiot introduced an algebraic approach to study linear differential equations based on the Galois theory for polynomials, see [8–13]. Combination of dynamical systems with differential Galois theory is a recent topic which started with the works of J.J Morales-Ruiz (see [12] and references therein) and with the works of J.-A. Weil (see [14]). Further works about applications of differential Galois theory include [15–17].

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The *Handbook of Exact Solutions of Ordinary Differential Equations*, see [18], is one important reference for scientists and engineers interested in solving explicitly ordinary differential equations. This book contains around 3,000 nonlinear ordinary differential equations with solutions, as well as exact, symbolic, and numerical methods for solving nonlinear equations. Nonlinear equations and systems with first-, second-, third-, fourth-, and higher-order are considered there.

Inspired by a previous version of the paper [19], we analysed the Exercise 11 in [18, §1.3.3], which corresponds to a five parametric family of differential equations. We discovered a typo (also corrected by us), which was corrected in the final version of [19] to study from differential Galois Theory point of view the integrability of the dynamical system proposed in such excercise.

We call as *Polyanin-Zaitsev vector field* to the vector field associated to this system of differential equations that comes from the corrigendum of the Exercise 11 in [18, §1.3.3]. Moreover, we study integrability aspects using differential Galois theory, following [9, 19] as well qualitative aspects due to the foliation associated to Polyanin-Zaitsev vector field is a Liénard equation, which is closely related to a Van Der Pol equation.

This paper not only present the corrigendum and complete solution of the Polyanin-Zaitsev excercise mentioned above, it also extends the results given in [19] concerning the Polyanin-Zaitsev vector field. From algebraic point of view we give conditions over the parameters to have polynomial vector field, moreover we obtain the critical points for some particular cases and we describe their behavior.

The results of this paper were obtained, but not published, during the seminar *Algebraic Methods in Dynamical Systems* in 2013 developed by the first author and in the master thesis of the second author in 2014 (supervised by the first and third author).

1 Preliminaries

In this section we provide the necessary theoretical background to understand the rest of the paper.

A *planar polynomial system* of degree *n* is given by

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y),$$

(1)

being $P, Q \in \mathbb{C}[x, y]$ and $n = \max(\deg P, \deg Q)$. By $X := (P, Q)$ we denote the *polynomial vector field* associated to the system (1). The planar polynomial vector field X can be also writen in the form

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

A *foliation* of a polynomial vector field of the form (1) is given by

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.$$

Following [6], we present the following theorem, which allow us the characterization of the critical points.

Theorem 1.1. *Let X, Y be analytic functions with polynomial part containing terms of degree greater than 1. Consider the planar differential system*

$$\dot{x} = y + X(x, y)$$
$$\dot{y} = Y(x, y)$$

being the origin an isolated critical point. Assume

$$y = F(x) = a_2 x^2 + a_3 x^3 + \ldots$$

as the solutions of $y + X(x, y) = 0$ near to $(0, 0)$. Suppose

$$f(x) = Y(x, F(x)) = ax^{\alpha}(1 + \ldots), \quad a \neq 0, \quad \alpha \geq 2$$

and also

$$\Phi(x) = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) |_{(x,F(x))} = bx^{\beta}(1 + \ldots), \quad b \neq 0, \quad \beta \geq 1.$$

Then the following statements hold:

a) *If α is even and $\alpha > 2\beta + 1$, then $(0, 0)$ is a saddle node.*
   If α is even and $\alpha < 2\beta + 1$ or $\Phi(x) \equiv 0$, then the flow near of $(0, 0)$ have two hyperbolic sectors.
b) *If α is odd and $a > 0$, then $(0, 0)$ is a saddle point.*
c) *If α is odd and $a < 0$, several cases can occur:*

  - c_1 $\alpha > 2\beta + 1,$ β *even*
  - c_2 $\alpha = 2\beta + 1,$ β *even*, $b^2 + 4a(\beta + 1) \geq 0,$

in this case for $b < 0$ the critical point $(0, 0)$ is a stable node, while for $b > 0$ the critical point $(0, 0)$ is unstable node.

  - c_3 $\alpha > 2\beta + 1,$ β *odd*
  - c_4 $\alpha = 2\beta + 1,$ β *odd*, $b^2 + 4a(\beta + 1) \geq 0,$

in this case the flow near to the critical point $(0, 0)$ is topologically conformed by an elliptic sector joint with an hyperbolic sector.

  - c_5 $\alpha = 2\beta + 1$ y $b^2 + 4a(\beta + 1) < 0$
  - c_6 $\alpha < 2\beta + 1,$ *or*, $\Phi(x) \equiv 0,$

in this case the critical point $(0, 0)$ is focus or center.

2 Corrigendum to the problem

The original Exercise 11, section 1.3.3 of the book of Polyanin-Zaitsev (see [18, §1.3.3.11]) was presented as follows:

$$yy' = (a(2m + k)x^{2k} + b(2m - k))x^{m-k-1})y - (a^2 m x^{4k} + c x^{2k} + b^2 m)x^{2m-2k-1}.$$

The transformation $z = x^k$, $y = x^m(t + ax^k + bx^{-k})$ leads to a Riccati equation with respect to $z = z(t)$:

$$(-mt^2 + 2mab - c)\frac{dz}{dt} = bk + tkz + akz^2. \tag{2}$$

The substitution $z = \frac{mt^2 + c_0}{ak} \frac{w'_t}{w}$, where $c_0 = c - 2abm$, reduces equation (2) to a second order linear equation:

$$(mt^2 + c_0)^2 w''_{tt} + (2m + k)t(mt^2 + c_0)w'_t + abk^2 w = 0. \tag{3}$$

The transformation $\xi = \frac{t}{\sqrt{t^2 + c_0/m}}$, $u = (1 - \xi^2)^{\mu/2}w$ where $\mu = -\frac{m+k}{2m}$ bring equation (3) to the Legendre equation 2.1.2.226:

$$(1 - \xi^2)u''_{\xi\xi} - 2\xi u'_{\xi} + [\nu(\nu + 1) - \mu^2(1 - \xi^2)^{-1}]u = 0$$

where ν is a root of the quadratic equation $\nu^2 + \nu + \frac{m^2 - k^2}{4m^2} - \frac{abk^2}{mc_0} = 0$.

A typo in this exercise does not allows its solving. The correction of the problem is presented in the following proposition:

Proposition 2.1. *Given the family of Liénard equations of the form:*

$$yy' = (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2 m x^{4k} + c x^{2k} + b^2 m)x^{2m-2k-1},$$

the change of variables $z = x^k$ and $y = x^m(t + ax^k + bx^{-k})$ allow to transform any equation of this family to a Riccati equation.

Proof. The system of equations, associated with this Liénard equation is:

$$\dot{x} = y$$

$$\dot{y} = (a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}.$$

Now, applying the transformation

$$z = x^k, \qquad y = az^{\frac{m+k}{k}} + tz^{\frac{m}{k}} + bz^{\frac{m-k}{k}}$$

and differentiating we have that:

$$dx = \frac{1}{k}z^{\frac{1-k}{k}}\,dz, \qquad dy = z^{\frac{m}{k}}\,dt + (t\frac{m}{k}z^{\frac{m-k}{k}} + a\frac{m+k}{k}z^{\frac{m}{k}} + b\frac{m-k}{k}z^{\frac{m-2k}{k}})dz,$$

then, the associated foliation has the form $ydy = (f(x)y - g(x))dx$, being $f(x) = (a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1})$ and $g(x) = (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}$ as the Liénard equations. Now we compute each part of this equality, thus we obtain the left side as:

$$kydy = k(az^{\frac{m+k}{k}} + tz^{\frac{m}{k}} + bz^{\frac{m-k}{k}})z^{\frac{m}{k}}\,dt + k(t^2\frac{m}{k}z^{\frac{2m-k}{k}} + at\frac{m+k}{k}z^{\frac{2m}{k}} bt\frac{m-k}{k}z^{\frac{2m-2k}{k}} + a^2\frac{m+k}{k}z^{\frac{2m+k}{k}} + at\frac{m}{k}z^{\frac{2m}{k}} + ab\frac{m-k}{k}z^{\frac{2m-k}{k}}$$

$$+tb\frac{m}{k}z^{\frac{2m-2k}{k}} + ab\frac{m+k}{k}z^{\frac{2m-k}{k}} + b^2\frac{m-k}{k}z^{\frac{2m-3k}{k}})dz$$

$$= (akz^2 + tkz + bk)z^{\frac{2m-k}{k}}\,dt + (t^2mz^{\frac{2m-k}{k}} + at(m+k)z^{\frac{2m}{k}} + bt(m-k)z^{\frac{2m-2k}{k}} + a^2(m+k)z^{\frac{2m+k}{k}}$$

$$+atmz^{\frac{2m}{k}} + ab(m-k)z^{\frac{2m-k}{k}} + tbmz^{\frac{2m-2k}{k}} + ab(m+k)z^{\frac{2m-k}{k}} + b^2(m-k)z^{\frac{2m-3k}{k}})dz$$

and the right side as

$$(f(x)y - g(x))dx = ((a(2m+k)x^{m+k-1} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1})dx$$

$$= ((a(2m+k)z^{\frac{m+k-1}{k}} + b(2m-k)z^{\frac{m-k-1}{k}})(az^{\frac{m+k}{k}} + tz^{\frac{m}{k}} + bz^{\frac{m-k}{k}})$$

$$-(a^2mz^k + cz^4 + b^2m)z^{\frac{2m-2k-1}{k}})\frac{1}{k}z^{\frac{1-k}{k}}\,dz$$

$$= ((a(2m+k)z^{\frac{m+k-1-(k-1)}{k}} + b(2m-k)z^{\frac{m-k-1-(k-1)}{k}})(az^{\frac{m+k}{k}} + tz^{\frac{m}{k}} + bz^{\frac{m-k}{k}})z^{\frac{k-1}{k}}$$

$$-(a^2mz^k + cz^4 + b^2m)z^{\frac{2m-2k-1}{k}})\frac{1}{k}z^{\frac{1-k}{k}}\,dz$$

$$= ((a(2m+k)z^{\frac{m}{k}} + b(2m-k)z^{\frac{m-2k}{k}})(az^{\frac{m+k}{k}} + tz^{\frac{m}{k}} + bz^{\frac{m-k}{k}})z^{\frac{k-1}{k}}$$

$$-(a^2mz^k + cz^4 + b^2m)z^{\frac{2m-2k-1}{k}})\frac{1}{k}z^{\frac{1-k}{k}}\,dz$$

$$= (at(2m+k)z^{\frac{2m}{k}} + a^2(2m+k)z^{\frac{2m+k}{k}} + ab(2m+k)z^{\frac{2m-k}{k}} + bt(2m-k)z^{\frac{2m-2k}{k}}$$

$$+ab(2m-k)z^{\frac{2m-k}{k}} + b^2(2m-k)z^{\frac{2m-3k}{k}} - a^2mz^{\frac{2m+k}{k}} - cz^{\frac{2m-k}{k}} - b^2mz^{\frac{2m-3k}{k}})\frac{1}{k}dz.$$

For our purpose, we organize the terms with respect to dz, that is:

$$(at(2m+k)z^{\frac{2m}{k}} + a^2(2m+k)z^{\frac{2m+k}{k}} + ab(2m+k)z^{\frac{2m-k}{k}} + bt(2m-k)z^{\frac{2m-2k}{k}}$$

$$+ab(2m-k)z^{\frac{2m-k}{k}} + b^2(2m-k)z^{\frac{2m-3k}{k}} - a^2mz^{\frac{2m+k}{k}} - cz^{\frac{2m-k}{k}} - b^2mz^{\frac{2m-3k}{k}}$$

$$-t^2mz^{\frac{2m-k}{k}} - at(m+k)z^{\frac{2m}{k}} - bt(m-k)z^{\frac{2m-2k}{k}} - a^2(m+k)z^{\frac{2m+k}{k}} - atmz^{\frac{2m}{k}}$$

$$-ab(m-k)z^{\frac{2m-k}{k}} - tbmz^{\frac{2m-2k}{k}} - ab(m+k)z^{\frac{2m-k}{k}} - b^2(m-k)z^{\frac{2m-3k}{k}})dz$$

$$= (akz^2 + tkz^+ bk)z^{\frac{2m-k}{k}}\,dt$$

Now organizing the terms again we have:

$$((ab(2m+k) + ab(2m-k) - c - t^2m - ab(m-k) - ab(m+k))z^{\frac{2m-k}{k}}$$

$$+(a^2(2m+k) - a^2m - a^2(m+k))z^{\frac{2m+k}{k}} + (at(2m+k) - at(m+k) - atm)z^{\frac{2m}{k}}\frac{dz}{dt}$$

$$= (akz^2 + tkz + bk)$$

Thus, we obtain the Riccati equation:

$$(-mt^2 + 2mab - c)\frac{dz}{dt} = bk + tkz + akz^2 \tag{4}$$

and we conclude the proof. □

For the rest of transformations proposed in the Exercise of Polyanin-Zaitsev we need the results concerning the transformations, which will be given in the next section.

3 Some transformations

In this section we study some transformations that allow us to complete the exercise stated by Polyanin-Zaitsev above.

Lemma 3.1. *If $R = a_1 x^2 + a_1 x + a_0$, $S = b_1 x + b_0$, with $a_2, b_1 \neq 0$. Then the differential equation*

$$R^2 \partial_x^2 y + SR \partial_x y + Cy = 0,$$

with $y = y(x)$ is transformed in the equation

$$Q^2 \partial_\tau^2 \hat{y} + LQ \partial_\tau \hat{y} + \lambda \hat{y} = 0,$$

with

$$Q = \tau^2 + q_0, \quad L = l_1 \tau + l_0, \quad \tau = x + \frac{a_1}{2a_2} \quad \text{and} \quad \hat{y} = y(x(\tau)).$$

Proof. $R^2 \partial_x^2 y + SR \partial_x y + Cy = 0$ then replacing

$$(a_1 x^2 + a_1 x + a_0)^2 \partial_x^2 y + (b_1 x + b_0)(a_1 x^2 + a_1 x + a_0) \partial_x y + Cy = 0.$$

We divide all by a_2^2, thus we get:

$$\left(x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2} \right)^2 \partial_x^2 y + \left(\frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left(x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2} \right) \partial_x y + \frac{C}{a_2^2} y = 0.$$

Now in R we complete the square

$$\left(x^2 + \frac{a_1}{a_2} x + \left(\frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2 \right)^2 \partial_x^2 y$$

$$+ \left(\frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left(x^2 + \frac{a_1}{a_2} x + \left(\frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2 \right) \partial_x y + \frac{C}{a_2^2} y = 0.$$

Then:

$$\left(\left(x + \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2 \right)^2 \partial_x^2 y$$

$$+ \left(\frac{b_1}{a_2} x + \frac{b_0}{a_2} \right) \left(\left(x + \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2 \right) \partial_x y + \frac{C}{a_2^2} y = 0.$$

Assume

$$\tau = x + \frac{a_1}{2a_2}, \quad q_0 = \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2,$$

then

$$\left(x + \frac{a_1}{2a_2} \right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2} \right)^2 = \tau^2 + q_0 = Q(\tau).$$

Replacing x in the polynomial S in term of τ, we get:

$$\frac{b_1}{a_2} x + \frac{b_0}{a_2} = \frac{b_1}{a_2} \left(\tau - \frac{a_1}{2a_2} \right) + \frac{b_0}{a_2} = \frac{b_1}{a_2} \tau - \frac{b_1 . a_1}{2a_2^2} + \frac{b_0}{a_2}.$$

Then, if $l_1 = \frac{b_1}{a_2}$ and $l_0 = -\frac{b_1 . a_1}{2a_2^2} + \frac{b_0}{a_2}$, we get $l_1 \tau + l_0 = L(\tau)$.

Now, if $\lambda = \frac{C}{a_2^2}$, the differential equations will be

$$Q^2 \hat{\partial}_\tau^2 \hat{y} + LQ \hat{\partial}_\tau \hat{y} + \lambda \hat{y} = 0,$$

where $\hat{y} = y(x(\tau))$, and the transformation $\tau = x + \frac{a_1}{2a_2}$ send ∂_x on ∂_τ.   □

Remark 3.1. The differential equation of the form

$$(1 - x^2)\partial_x^2 y + (\check{b} - \tilde{a} - (\tilde{a} + \check{b} + 2)x)\partial_x y + \lambda y = 0,$$

with $\lambda = n(n + 1 + \tilde{a} + \check{b})$ and $n \in \mathbb{N}$, is known as Jacobi equation (in general form). It is a particular case of the hypergeometric equation, but the solutions include Jacobi polynomials. If we take $\tilde{a} = \check{b}$ and $\tilde{\lambda} = n(n + 2\tilde{a})$ with $n \in \mathbb{N}$, we get a Gegenbauer equation (or ultraspherical case):

$$(1 - x^2)\partial_x^2 y - 2(\tilde{a} + 1)x\partial_x y + \tilde{\lambda}y = 0.$$

Now we study a special transformation, in the following theorem:

Theorem 3.2. *The differential equation*

$$a_n z^{(n)} + (\tilde{k} + a_{n-1})z^{(n-1)} + a_{n-2}z^{(n-2)} + \ldots + a_1 z^{(1)} + a_0 z = 0,$$

with $a_i \in \mathbb{C}(x)$, with $a_n \neq 0$, can be transformed into the differential equation

$$y^{(n)} + \tilde{k}y^{(n-1)} + b_{n-2}y^{(n-2)} + b_{n-3}y^{(n-3)} + \ldots + b_1 y^{(1)} + b_0 y = 0$$

with $b_i \in \mathbb{C}(x)$.

Proof. Following the Lemma 3.1 and taking the implicit transformation $z = \varepsilon(t)y + \mu(t)$ with $\varepsilon(t), \mu(t) \in \mathbb{C}(x)$, we compute the first equation applying the change of variable

$$(\varepsilon y)^{(n)} + \mu^{(n)} + \Sigma_{i=0}^{n-2} b_i(\varepsilon y + \mu)^{(i)} = 0.$$

Then, computing in general form the Leibniz rule we get:

$$\varepsilon^{(n)}y + \binom{n-1}{1}(\varepsilon)^{(n-1)}y^{(1)} + \binom{n}{2}(\varepsilon)^{(n-2)}y^{(2)} + \binom{n}{3}(\varepsilon)^{(n-3)}y^{(3)} + \ldots \binom{n}{n-1}\varepsilon^{(1)}y^{(n-1)} + \binom{n}{n}(\varepsilon)y^{(n)}$$

$$+ (\tilde{k} + a_{n-1})\Big[\varepsilon^{(n-1)}y + \binom{n-1}{1}(\varepsilon)^{(n-2)}y^{(1)} + \binom{n-1}{2}(\varepsilon)^{(n-3)}y^{(2)} + \binom{n-1}{3}(\varepsilon)^{(n-4)}y^{(3)}$$

$$+ \ldots \binom{n-1}{n-2}\varepsilon^{(1)}y^{(n-2)} + \binom{n-1}{n-1}(\varepsilon)y^{(n-1)}\Big] + a_{n-2}\Big[\varepsilon^{(n-2)}y + \binom{n-2}{1}(\varepsilon)^{(n-3)}y^{(1)}$$

$$+ \binom{n-2}{2}(\varepsilon)^{(n-4)}y^{(2)} + \binom{n-2}{3}(\varepsilon)^{(n-5)}y^{(3)} + \ldots \binom{n-2}{n-3}\varepsilon^{(1)}y^{(n-3)} + \binom{n-2}{n-2}\varepsilon y^{(n-2)}\Big]$$

$$+ a_{n-3}\Big[\varepsilon^{(n-3)}y + \binom{n-3}{1}(\varepsilon)^{(n-4)}y^{(1)} + \binom{n-3}{2}(\varepsilon)^{(n-5)}y^{(2)} + \binom{n-3}{3}(\varepsilon)^{(n-6)}y^{(3)}$$

$$+ \ldots \binom{n-3}{n-4}\varepsilon^{(1)}y^{(n-4)} + \binom{n-4}{n-4}\varepsilon y^{(n-3)}\Big] + \ldots a_{n-m}\Big[\varepsilon^{(n-m)}y + \binom{n-m}{1}(\varepsilon)^{(n-m-1)}y^{(1)}$$

$$+ \binom{n-m}{2}(\varepsilon)^{(n-m-2)}y^{(2)} + \binom{n-m}{3}(\varepsilon)^{(n-m-3)}y^{(3)} + \ldots \binom{n-m}{n-m-1}\varepsilon^{(1)}y^{(n-m-1)} + \binom{n-m}{n-m}\varepsilon y^{(n-m)}\Big] + \ldots = 0$$

Continuing of this form, we divide all equation by ε. Then, we use the same method of the indeterminate coefficients to calculate ε and the b_i coefficient:
If we take $y^{(n-1)}$:

$$\binom{n}{n-1}\frac{\varepsilon^{(1)}}{\varepsilon} + \tilde{k} + a_{(n-1)} = \tilde{k}$$

then:

$$n\frac{\varepsilon^{(1)}}{\varepsilon} + a_{(n-1)} = 0$$

From this differential equation we get an appropriate ε value, and with it we obtain the coefficient b_i.

If we take $y^{(n-2)}$:

$$\binom{n}{n-2}\frac{\varepsilon^{(2)}}{\varepsilon} + a_{(n-1)}\binom{n-1}{n-2}\frac{\varepsilon^{(1)}}{\varepsilon} + a_{n-2}\binom{n-2}{n-2} = b_{n-2}$$

If we take $y^{(n-3)}$:

$$\binom{n}{n-3}\frac{\varepsilon^{(3)}}{\varepsilon} + a_{n-1}\binom{n-1}{n-3}\frac{\varepsilon^{(2)}}{\varepsilon} + a_{n-2}\binom{n-2}{n-3}\frac{\varepsilon^{(1)}}{\varepsilon} + a_{n-3}\binom{n-3}{n-3} = b_{n-3}$$

Continuing of this form we see, that for any $k \in \mathbb{N}$ the recurrent formulae will be:

$$\binom{n}{n-k}\frac{\varepsilon^{(k)}}{\varepsilon} + \dots + a_{n-k+3}\binom{n-k+3}{n-k}\frac{\varepsilon^{(3)}}{\varepsilon} + a_{n-k+2}\binom{n-k+2}{n-k}\frac{\varepsilon^{(2)}}{\varepsilon} + a_{n-k+1}\binom{n-k+1}{n-k}\frac{\varepsilon^{(1)}}{\varepsilon} + a_{n-k}\binom{n-k}{n-k} = b_{n-k} \qquad \square$$

If in the previous theorem, we consider $\tilde{k} = 0$, we get the next corollary:

Corollary 3.3. *The differential equation*

$$a_n z^{(n)} + a_{n-1} z^{(n-1)} + a_{n-2} z^{(n-2)} + \dots + a_1 z^{(1)} + a_0 z = 0$$

with $a_i \in \mathbb{C}(x)$ can be transformed in the following equation

$$y^{(n)} + b_{n-2} y^{(n-2)} + b_{n-3} y^{(n-3)} + \dots + b_1 y^{(1)} + b_0 y = 0,$$

where $b_i \in \mathbb{C}(x)$.

Example: Applying the transformation over the general second order differential equation $z'' + a_1 z' + a_0 z = 0$:

Using Theorem 3.2, we get $\frac{2\varepsilon'}{\varepsilon} + a_1 = 0$, then $\varepsilon' = -\frac{a_1}{2}\varepsilon$. Now, through derivatives and dividing by ε, we get $\varepsilon'' = -\frac{a_1'}{2} - \frac{\varepsilon'}{\varepsilon}\frac{a_1}{2}$, but $\frac{\varepsilon'}{\varepsilon} = -\frac{a_1}{2}$. Thus, we obtain

$$\varepsilon'' = -\frac{a_1'}{2} + \frac{a_1^2}{4}.$$

In this way we arrive to $b = -\frac{a_1'}{2} + \frac{a_1^2}{4}$, for the differential equation $y'' + by = 0$.

In the following theorem we recall that a Hamiltonian Change of Variable $z = z(x)$ is a change of variable, where $(z(x), z'(x))$ is a solution curve of a Hamiltonian system of one degree of freedom. The new derivation is given by $\hat{\partial}_z = \sqrt{\alpha}\partial_z$, being $\alpha = (\partial_x z)^2$, see [15–17] and references therein.

Theorem 3.4. *Let Q and L be as in (3.1), with $a_1 = b_1 = 0$ or $b_0 = \frac{a_1 b_1}{2a_2}$. Through the Hamiltonian change of variable $\xi = \partial_\tau \sqrt{Q}$, the differential equation*

$$Q^2 \partial_t^2 w + LQ\partial_t w + \lambda w = 0,$$

is transformed in the equation

$$(1 - \xi^2)\partial_\tau^2 \hat{w} + \left(l_1 - \frac{3}{q_0}\right)\xi\partial_\tau \hat{w} + \frac{\lambda}{q_0}\hat{w} = 0.$$

Owing to $\lambda = n(n + 1 + \tilde{a} + \tilde{b})$, we have the Jacobi equation with $\tilde{a} = \tilde{b}$. Moreover, if $\lambda = n(n + 2\tilde{a})$ then we obtain a Gegenbauer equation.

Proof. **Case 1:**

If we assume $a_1 = b_0 = 0$, we obtain $l_0 = 0$, $l_1 = \frac{b_1}{a_2}$, $q_0 = \frac{a_0}{a_2}$. Then:

$$(\tau^2 + q_0)^2 \partial_\tau^2 w + l_1 \tau (\tau^2 + q_0) \partial_\tau w + \lambda w = 0.$$

Now, by hypothesis $\xi = \partial_\tau \sqrt{Q}$, that is

$$\xi = \frac{\partial_\tau Q}{2\sqrt{Q}} = \frac{\partial_\tau (\tau^2 + q_0)}{2\sqrt{\tau^2 + q_0}},$$

thus

$$\xi^2 = \frac{\tau^2 + q_0}{\tau^2 + q_0} - \frac{q_0}{\tau^2 + q_0} \frac{q_0}{1 - \xi^2} = \tau^2 + q_0 \tau^2 = \frac{q_0}{1 - \xi^2} - q_0 \tau = \pm \frac{\xi \sqrt{q_0}}{\sqrt{1 - \xi^2}}.$$

Furthermore, due to $\xi = \partial_\tau \sqrt{Q}$ then α we arrive to:

$$\sqrt{\alpha} = \partial_\tau \xi = \partial_\tau^2 \sqrt{Q} = \frac{1}{\frac{\sqrt{q_0}(\sqrt{1-\xi^2} - \xi \cdot \frac{2\xi}{2\sqrt{1-\xi^2}})}{1-\xi^2}} = \frac{1}{\frac{\sqrt{q_0}(1-\xi^2+\xi^2)}{\sqrt{1-\xi^2}(1-\xi^2)}} = \sqrt{\frac{(1-\xi^2)^3}{q_0}},$$

i.e $\alpha = \frac{(1-\xi^2)^3}{q_0}$.

Now $\hat{\partial}_\xi = \sqrt{\frac{(1-\xi^2)^3}{q_0}} \partial_\xi$, and $\hat{\partial}_\xi^2 = \alpha \partial_\xi^2 + \frac{1}{2} \partial_\xi \alpha \partial_\xi = \frac{(1-\xi^2)^3}{q_0} \partial_\xi^2 - \frac{3(1-\xi^2)^2}{q_0} \partial_\xi$.

We compute all elements of the Hamiltonian change of variable $\hat{Q} = Q(\tau(\xi)) = \frac{q_0^2}{1-\xi^2}$, $\hat{L} = l_1 \frac{\xi\sqrt{q_0}}{\sqrt{1-\xi^2}}$, $\hat{w}(\xi(\tau)) = w(\tau)$. Then:

$$Q^2 \partial_\tau^2 w \rightsquigarrow \hat{Q}^2 \hat{\partial}_\xi^2 \hat{w} = \left(\frac{q_0^2}{1-\xi^2}\right)^2 \left(\frac{(1-\xi^2)^3}{q_0} \partial_\xi^2 \hat{w} - \frac{3(1-\xi^2)^2}{q_0} \partial_\xi \hat{w}\right) = q_0(1-\xi^2)\partial_\xi^2 \hat{w} - 3\xi \partial_\xi \hat{w},$$

$$LQ\partial_\tau w \rightsquigarrow \hat{L}\hat{Q}\hat{\partial}_\xi \hat{w} = \left(l_1 \frac{\xi\sqrt{q_0}}{\sqrt{1-\xi^2}}\right)\left(\frac{q_0^2}{1-\xi^2}\right)\sqrt{\frac{(1-\xi^2)^3}{q_0}}\partial_\xi \hat{w} = l_1 q_0 \xi \partial_\xi \hat{w}.$$

If we replace, in the transformed differential equation, we obtain:

$$q_0(1-\xi^2)\partial_\xi^2 \hat{w} + (l_1 q_0 - 3)\xi \partial_\xi \hat{w} + \lambda \hat{w} = 0.$$

It is equivalent to Gegenbauer equation with $\tilde{\lambda} = \frac{\lambda}{q_0}$, $-2\tilde{a} + 1 = l_1 - \frac{3}{q_0}$ i.e $\tilde{a} = \frac{\frac{3}{q_0} - l_1 - 1}{2}$ then:

$$(1-\xi^2)\partial_\xi^2 \hat{w} + \left(l_1 - \frac{3}{q_0}\right)\xi \partial_\xi \hat{w} + \frac{\lambda}{q_0}\hat{w} = 0.$$

Finally if $\frac{\lambda}{q_0} = n(n + \frac{3}{q_0} - l_1 - 1)$ with $n \in \mathbb{N}$, then the solutions of this equation are Ultraspherical Gegenbauer polynomial.

Case 2:

If $b_0 = \frac{a_1 b_1}{2a_2}$ then:

$$Q(\tau) = \tau + q_0 \text{ with } q_0 = \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2}\right)^2, L(\tau) = \tilde{l}_1 \tau + \tilde{l}_0, \tilde{l}_1 = \frac{b_1}{a_2} \text{ and } \tilde{l}_0 = -\frac{b_1 a_1}{2a_2^2} + \frac{b_0}{a_2} = -\frac{b_1 a_1}{2a_2^2} + \frac{b_1 a_1}{2a_2^2} = 0$$

i.e the initial differential equation will be:

$$(\tau^2 + q_0)^2 \partial_\tau^2 \hat{w} + \tilde{l}_1 \tau (\tau^2 + q_0)\partial_\tau \hat{w} + \lambda \hat{w} = 0$$

for instance, the same differential equation of the previous case. □

Now we transform the Gegenbauer equation into an Hypergeometric equation.

Lemma 3.5. *The Gegenbauer equation*

$$(1 - x^2)\partial_x^2 y - 2(\mu + 1)x\partial_x y + (\nu - \mu)(\nu + \mu + 1)y = 0$$

is transformed into an hypergeometric equation

$$z(1 - z)\partial_z^2 y + (c - (a + b + 1)z)\partial_z y - aby = 0,$$

where $a = \mu - \nu$, $b = \nu + \mu + 1$, $c = \mu + 1$ and $z = \frac{1-x}{2}$.

Proof. For this transformation we will use a Hamiltonian change of variable, over the independent variable of the Gegenbauer equation $x = 1 - 2z$. Then $\sqrt{\alpha} = \partial_x z = -\frac{1}{2}$, that is, $\alpha = \frac{1}{4}$, $\hat{\partial}_z = \frac{1}{2}\partial_z$ and $\hat{\partial}_z^2 = \frac{1}{4}\partial_z^2$.

Substituting in the Gegenbauer equation, we obtain

$$(1 - x^2)\partial_x^2 y = (1 - (1 - 4z + 4z^2))\frac{1}{4}\hat{\partial}_z^2\hat{y} = z(1 - z)\hat{\partial}_x^2\hat{y}$$

Thus, we obtain the equation

$$z(1 - z)\hat{\partial}_x^2\hat{y} + (\mu + 1)(1 - 2z)\hat{\partial}_x\hat{y} - (\mu - \nu)(\nu + \mu + 1)\hat{y} = 0.$$

We know that Hypergeometric equation is of the form

$$z(1 - z)\partial_z^2 y + (c - (a + b + 1)z)\partial_z y - aby = 0.$$

Then, we compute the parameters values a, b and c as follows:

$$ab = (\mu - \nu)(\mu + \nu + 1),$$

$$c - (a + b + 1)z = (\mu + 1) - 2(\mu + 1)z.$$

Therefore

$$c = \mu + 1, \quad a = \mu - \nu, \quad b = \mu + \nu + 1,$$

which concludes the proof. □

Proposition 3.6. *Through the Hamiltonian change of variables $x = 1 - 2\xi$ and $y = (x^2 - 1)^{\frac{\mu}{2}}w$, we can transform the Hypergeometric equation*

$$\xi(\xi - 1)\partial_\xi^2 w + (\mu + 1)(1 - 2\xi)\partial_\xi w + (\nu - \mu)(\nu + \mu + 1)w = 0$$

into the Legendre equation

$$(1 - x^2)\partial_x^2 y - 2x(1 - x^2)\partial_x y + [\nu(\nu + 1)(1 - x^2) - \mu^2]y = 0.$$

Proof. Firstly we transform the Legendre equation into the Hypergeometric equation:

If $y = (x^2 - 1)^{\frac{\mu}{2}}w$, then
$$\partial_x y = \mu x(x^2 - 1)^{\frac{\mu}{2}-1}w + (x^2 - 1)^{\frac{\mu}{2}}\partial_x w$$

and
$$\partial_x^2 y = \mu[(x^2 - 1)^{\frac{\mu}{2}-1} + 2(\frac{\mu}{2} - 1)x^2(x^2 - 1)^{\frac{\mu}{2}-2}]w + 2\partial_x(x^2 - 1)^{\frac{\mu}{2}-1}\partial_x w + (x^2 - 1)^{\frac{\mu}{2}}\partial_x^2 w.$$

Now dividing by $(x^2 - 1)^{\frac{\mu}{2}-2}$ and replacing in each terms of the Legendre equation we get:

$$(x^2 - 1)^2\partial_x^2 y = \mu(x^2 - 1)w + 2\mu(\frac{\mu}{2} - 1)x^2 w + 2\mu x(x^2 - 1)\partial_x w + (x^2 - 1)^2\partial_x^2 w$$

$$2x(x^2 - 1)\partial_x y = 2\mu x^2 w + 2x(x^2 - 1)\partial_x w$$

$$[v(v + 1)(1 - x^2) - \mu^2]y = [-v(v + 1)(x^2 - 1) - \mu^2]w$$

Now we obtain

$$(x^2 - 1)^2 \partial_x^2 w + [2\mu x + 2x](x^2 - 1)\partial_x w + (-v(v + 1)(x^2 - 1) - \underbrace{\mu^2 + 2\mu x^2} + \mu(x^2 - 1) + 2\mu(\tfrac{\mu}{2} - 1)x^2)w = 0$$

$$\Rightarrow$$

$$(x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)(x^2 - 1)\partial_x w + (-v(v + 1)(x^2 - 1) - \underbrace{\mu^2 + 2\mu x^2 + \mu x^2 - \mu + \mu^2 x^2 - 2\mu x^2}) = 0$$

$$\Rightarrow$$

$$(x^2 - 1)^2 \partial_x^2 w + 2x(\mu + 1)(x^2 - 1)\partial_x w + (-v(v + 1)(x^2 - 1) + \mu^2(x^2 - 1) + \mu(x^2 - 1))w = 0$$

$$\Rightarrow$$

$$(x^2 - 1)\partial_x^2 w + 2x(\mu + 1)\partial_x w + [\mu^2 - v^2 + \mu - v]w = 0$$

$$\Rightarrow$$

$$(x^2 - 1)\partial_x^2 w + 2x(\mu + 1)\partial_x w + (\mu - v)(\mu + v + 1)w = 0$$

Now applying the Hamiltonian change of variable $x = 1 - 2\xi$ $\xi = \frac{1-x}{2}$, we obtain $\partial_x \xi = \frac{-1}{2} = \sqrt{\alpha}$, being $\alpha = \frac{1}{4}$. Thus, we obtain $\hat{\partial}_\xi = \frac{-1}{2}\partial_\xi$, then $x^2 - 1 = 1 - 4\xi + 4\xi^2 - 1 = 4\xi(\xi - 1)$. Now, replacing we obtain:

$$\xi(\xi - 1)\partial_\xi \hat{w} - (\mu + 1)(1 - 2\xi)\partial_\xi \hat{w} + (\mu - v)(\mu + v + 1)\hat{w} = 0$$

Following exactly the reversed process, we can transform a Hypergeometric equation into Legendre equation.

<div align="right">□</div>

Example. Transform the next equation on ultraespheric form.

$$(mt^2 + c_0)^2 \ddot{w} + t(2m + k)(mt^2 + c_0)\dot{w} + abk^2 w = 0.$$

Now $R = mt^2 + c_0$, $S = (2m + k)t$ $\tau = t + \frac{c_0}{2m}$, $q_0 = \frac{c_0}{m}$, $l_0 = 0$, $l_1 = \frac{2m+k}{m}$ and $\lambda = \frac{1}{a}bk^2 m^2$.

Applying the previous lemma we have the equation:

$$(\tau^2 + \frac{c_0}{m})^2 \partial_\tau^2 \hat{w} + \frac{2m + k}{m}\tau(\tau^2 + \frac{c_0}{m})\partial_\tau \hat{w} + \frac{1}{a}bk^2 m^2 \tilde{w} = 0.$$

Now applying the previous theorem we get:

$$(1 - \xi^2)\partial_\xi^2 \hat{u} + (\frac{2m + k}{m} - \frac{3m}{c_0})\xi\partial_\xi \hat{u} + \frac{abk^2}{c_0 m}\hat{u} = 0$$

with $\hat{u}(\xi) = \hat{w}(\tau(\xi))$.

Remark 3.2. If we have our equation in the Legendre form, we apply the proposition 3.6 and therefore we can study it as in [19] to conclude the integrability or non-integrability, of the Liénard equation. Moreover, such as we will see in the next section, through equation (5) in Legendre's form we can apply the Kimura table, see [19].

4 Polyanin-Zaitsev vector field

The associated system of the Polyanin-Zaitsev vector field, with $a, b, c, m, k \in \mathbb{R}$, is given by:

$$\dot{x} = y$$

$$(5)$$

$$\dot{y} = (\alpha x^{m+k-1} + \beta x^{m-k-1})y - \gamma x^{2m-2k-1}.$$

with $\alpha = a(2m + k)$, $\beta = b(2m - k)$ and $\gamma = (a^2 mx^{4k} + cx^{2k} + b^2 m)$, where the Polyanin-Zaitsev vector field is given by $X =: (P, Q)$, being,

$$(P, Q) := (y, (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2 mx^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1}).$$

The next proposition can illustrate the cases in which the Polyanin-Saitsev vector field is formed by non trivial polynomial functions.

Proposition 4.1. *The system* (5) *is a not null differential polynomial system if it is equivalently to one of the next families:*

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= [a\tfrac{3s+p+4}{2}x^s + b\tfrac{s+3p+4}{2}x^p]y - a\tfrac{s+p+2}{2}x^{2s+1} - cx^{s+p+1} - b^2\tfrac{s+p+2}{2}x^{2p+1}\end{aligned} \tag{6}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= b\tfrac{r+2p+3}{2}yx^p - cx^r - b^2\tfrac{r+1}{2}x^{2p+1}\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= [a\tfrac{3s+p+4}{2}x^s + b\tfrac{s+3p+4}{2}x^p]y - a\tfrac{s+p+2}{2}x^{2s+1} - b^2\tfrac{s+p+2}{2}x^{2p+1}\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -cx^{s+p+1}\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a\tfrac{3s+p+4}{2}yx^s - a\tfrac{r+1}{2}x^{2s+1} - cx^r\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= +b(m + p + 1)yx^p - b^2 mx^{2p+1}\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a(m + s + 1)yx^s - amx^{2s+1}\end{aligned} \tag{12}$$

with $s, p, r \in \mathbb{Z}$ *defined in the proof.*

Proof. The system (5) is a polynomial system if Q is a polynomial function, that is, the exponents of each term must be non negative integer. Furthermore, we need to consider the values of the constants a, b and c. Now we consider the different possibilities for these constants:

Case 1. For $a \neq 0$, $b \neq 0$, $c \neq 0$, it must be satisfied:

$$\begin{aligned}m + k - 1 &= s\\ m - k - 1 &= p\\ 2m + 2k - 1 &= 2s + 1\\ 2m - 1 &= r\\ 2m - 2k - 1 &= 2p + 1,\end{aligned}$$

being $s, p, r \in \mathbb{Z}^+$. Thus $m = \frac{r+1}{2}$ and $k = \frac{s-p}{2}$, which means that $r = s + p + 1$. Therefore we obtain the following system associated with the family (5):

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= [a\tfrac{3s+p+4}{2}x^s + b\tfrac{s+3p+4}{2}x^p]y - a\tfrac{s+p+2}{2}x^{2s+1} - cx^{s+p+1} - b^2\tfrac{s+p+2}{2}x^{2p+1}.\end{aligned}$$

Case 2. For $a = 0$, $b \neq 0$, $c \neq 0$, the system (5) is reduced to:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= b(2m - k)yx^{m-k-1} - b^2 mx^{2m-2k-1} - cx^{2m-1},\end{aligned}$$

since the exponents must be non-negative integers, then:

$$m - k - 1 = p$$
$$2m - 1 = r$$
$$2m - 2k - 1 = 2p + 1,$$

for instance, we arrive to the system:

$$\dot{x} = y$$
$$\dot{y} = b\frac{r+2p+3}{2}yx^p - cx^r - b^2\frac{r+1}{2}x^{2p+1}.$$

Case 3. For $a \neq 0$, $b \neq 0$, $c = 0$, the system (5) is reduced to:

$$\dot{x} = y$$

$$\dot{y} = a(2m + k)yx^{m+k-1} + b(2m - k)yx^{m-k-1} - a^2mx^{2m+2k-1} - b^2mx^{2m-2k-1},$$

again the exponents must be non-negative integers, therefore:

$$m + k - 1 = s$$
$$m - k - 1 = p$$
$$2m + 2k = 2s + 1$$
$$2m - 2k - 1 = 2p + 1,$$

thus, $m = \frac{s+p+2}{2}$ and $k = \frac{s-p}{2}$, which lead us to the following system:

$$\dot{x} = y$$
$$\dot{y} = b\frac{s+3p+4}{2}yx^p - cx^r - b^2\frac{r+1}{2}x^{2p+1}.$$

Case 4. For $a = 0$, $b = 0$, $c \neq 0$, the system (5) is reduced to:

$$\dot{x} = y$$

$$\dot{y} = cx^{2m-1},$$

due to the exponents must be non-negative integers, we arrive to $2m - 1 = r \in \mathbb{N}$, that is, $m = \frac{r+1}{2}$.

$$\dot{x} = y$$
$$\dot{y} = -cx^{s+p+1}.$$

Case 5. For $a \neq 0$, $b = 0$, $c \neq 0$, the system (5) is reduced to:

$$\dot{x} = y$$

$$\dot{y} = a(2m + k)yx^{m+k-1} - a^2mx^{2m+2k-1} - cx^{2m-1},$$

in this case:

$$m + k - 1 = s$$
$$2m + 2k - 1 = 2s + 1$$
$$2m - 1 = r,$$

then $m = \frac{r+1}{2}$ and $k = \frac{2s-r+1}{2}$. Thus we arrive to the system:

$$\dot{x} = y$$
$$\dot{y} = a\frac{3s+p+4}{2}yx^s - a\frac{r+1}{2}x^{2s+1} - cx^r.$$

Case 6. For $a = 0$, $b \neq 0$, $c = 0$, the system (5) is reduced to:

$$\dot{x} = y$$

$$\dot{y} = b(2m - k)yx^{m-k-1} - b^2 mx^{2m-2k-1},$$

in this case:

$$m - k - 1 = p$$
$$2m - 2k - 1 = 2p + 1,$$

then there is a line of solutions, with $r \in \mathbb{Z}^+$.
In this case the associated family is:

$$\dot{x} = y$$
$$\dot{y} = +b(m + p + 1)yx^p - b^2 mx^{2p+1}.$$

Case 7. For $a \neq 0$, $b = 0$, $c = 0$, the system (5) is reduced to:

$$\dot{x} = y$$

$$\dot{y} = a(2m + k)yx^{m+k-1} - a^2 mx^{2m+2k-1},$$

in this case:

$$m + k - 1 = s$$
$$2m + 2k - 1 = 2s + 1,$$

then the associated family is:

$$\dot{x} = y$$
$$\dot{y} = a(m + s + 1)yx^s - amx^{2s+1}.$$

$\square$

5 Finite critical points

In this section, we present an study about the existence of finite critical points and the stability for each family associated to the Polyanin-Zaitsev vector field.

Proposition 5.1. *For the vector field given by* (5), *with $k \in \mathbb{Z}$ the following statements holds:*

    a. *If $c^2 - 4a^2 b^2 m^2 < 0$, or $c^2 - 4a^2 b^2 m^2 \geq 0$ and $c > 0$, then $(0, 0)$ is the only one finite critical point of the family.*
    b. *If $c^2 - 4a^2 b^2 m^2 = 0$ and $c < 0$ then the system have three finite critical points.*
    c. *If $c^2 - 4a^2 b^2 m^2 \geq 0$, $c < 0$ and $k \in \mathbb{Z}$ then exist five finite critical points for the systems.*

Proof. For this proof we take the family (6) as form (5). Then, we will have to solve the system:

$$y = 0$$

$$(a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2 mx^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1} = 0$$

If $y = 0$, then $(a^2 mx^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1} = 0$, with $2m - 2k - 1 \geq 1$, then for the product will be equal to 0, must be $x = 0$ or $(a^2 mx^{4k} + cx^{2k} + b^2 m) = 0$. In the first case we obtain that $x = 0$ and we conclude $(x, y) = (0, 0)$.

For the second case, if we completing squares in the polynomial, we will have that:

$$a^2 m x^{4k} + c x^{2k} + b^2 m = a^2 m ((x^{2k} + \frac{c}{2a^2 m})^2 - (\frac{c^2 - 4a^2 b^2 m^2}{4a^4 m^2}))$$

$$= a^2 m (x^{2k} + \frac{c + \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m})(x^{2k} + \frac{c - \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m})$$

We can see that if $c^2 - 4a^2 b^2 m < 0$, then there are not real roots. That is, the only finite critical point is $(0, 0)$.

If $c^2 - 4a^2 b^2 m \geq 0$ and $c > 0$ for the equations we have that

$$(c + \sqrt{c^2 - 4a^2 b^2 m^2})(c - \sqrt{c^2 - 4a^2 b^2 m^2}) > 0.$$

That is $c - \sqrt{c^2 - 4a^2 b^2 m^2} < 0$ and $c + \sqrt{c^2 - 4a^2 b^2 m^2} < 0$ the previous to implies, there are not real roots. Then is, the only finite critical point is $(0, 0)$.

Now, if $c < 0$ and $c^2 - 4a^2 b^2 m = 0$, we have that

$$y = 0 \qquad \& \qquad (a^2 m x^{4k} + c x^{2k} + b^2 m) x^{2m - 2k - 1} = 0.$$

Again $(0, 0)$ is a first critical point. Therefore the equations $a^2 m x^{4k} + c x^{2k} + b^2 m = 0$, have real solutions. This is the system have critical points $(\pm \sqrt[2k]{\frac{-c}{2a^2 m}}, 0)$ and $(0, 0)$.

Now if $c < 0$ and $c^2 - 4a^2 b^2 m > 0$ the expressions,

$$x_1^k = \pm \sqrt{\frac{-c + \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m}}, \qquad x_2^k = \pm \sqrt{\frac{-c - \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m}},$$

are boot real. If we will to consider $k \in \mathbb{Z}$, then the critical points for (6) are

$$(0, 0), \qquad \left(\pm \sqrt[2k]{\frac{-c + \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m}}, 0 \right), \qquad \left(\pm \sqrt[2k]{\frac{-c - \sqrt{c^2 - 4a^2 b^2 m^2}}{2a^2 m}}, 0 \right).$$

$\square$

Proposition 5.2. *For the system* (7) *and* (10) *there are three critical points.*

Proof. If

$$y = 0$$
$$b^{\frac{s+3p+4}{2}} y x^p - c x^r - b^2 \frac{r+1}{2} x^{2p+1} = 0$$

then $\deg(Q) = max\{r, 2p + 1\}$, that is, we should consider two cases.

–   If $\deg(Q) = 2p + 1$ then $x^r (c + b^2 (\frac{r+1}{2}) x^{2p+1-r}) = 0$. This implies that

$$x = 0 \quad or \quad x^\gamma = \frac{-2c}{b^2(r + 1)}$$

being $\gamma = 2p - r + 1$. If γ is even then it is necessary that $c < 0$, and therefore the critical points are $(0, 0)$, $\left(\sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$ and $\left(- \sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$.
    If γ is odd then the critical points are $(0, 0)$ and $\left(\sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$.
–   If $\deg(Q) = r$ analogously $x^{2p+1} (c x^{r-2p-1} + \frac{b^2(r+1)}{2}) = 0$. If $r - 2p - 1 = \gamma$ is even then the critical points for the system (7) are $(0, 0)$, $\left(\sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$. On the other hand, if γ is odd, it is necessary that $c < 0$ and for instance the critical points are $(0, 0)$, $\left(\sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$ and $\left(- \sqrt[\gamma]{\frac{-2c}{b^2(r+1)}}, 0 \right)$.

Now for the family (10), we have that

$$y = 0$$
$$a\tfrac{3s+p+4}{2}yx^s - a\tfrac{r+1}{2}x^{2s+1} - cx^r = 0.$$

Then $\deg(Q) = max\{r, 2s + 1\}$, again we have to consider two cases.

– If $\deg(Q) = 2s + 1$ then $x^r\left(a\left(\tfrac{r+1}{2}\right)x^{2s+1-r} + c\right) = 0$. This implies that

$$x = 0 \quad or \quad x^\gamma = \frac{-2c}{a(r + 1)}, \quad \gamma = 2s - r + 1.$$

If γ is even then it is necessary that $c < 0$, and for instance the critical points are $(0, 0)$, $\left(\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$ and $\left(-\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$.

If γ is odd then the critical points are $(0, 0)$ and $\left(\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$.

– If $\deg(Q) = r$ analogously $x^{2s+1}\left(cx^{r-2s-1} + \tfrac{a(r+1)}{2}\right) = 0$. If $r - 2s - 1 = \gamma$ is even then the critical points for the system (10) are $(0, 0)$, $\left(\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$. On the other hand, if γ is odd, it is necessary that $c < 0$ and for instance the critical points are $(0, 0)$, $\left(\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$ and $\left(-\sqrt[\gamma]{\tfrac{-2c}{a(r+1)}}, 0\right)$.

$\square$

Proposition 5.3. *For systems of the form* (8), (9), (11) *and* (12), *the point* $(0, 0)$ *is the only critical point.*

Proof. We can see that the common characteristic in these families is that $c = 0$. Then for the family (8)

$$y = 0$$
$$\left(a\tfrac{3s+p+4}{2}x^s + b\tfrac{s+3p+4}{2}x^p\right)y - a\tfrac{s+p+2}{2}x^{2s+1} - b^2\tfrac{s+p+2}{2}x^{2p+1} = 0$$

Now we have two cases. If $\deg(Q) = 2s + 1$ then $x^{2p+1}\left(\tfrac{s+p+2}{2}\right)(x^{2s-2p} + b^2) = 0$, where $s, p \in \mathbb{Z}^+$ and $2(s - p)$ is even. Therefore, we can conclude that the only solution for the systems under the conditions given above is $(0, 0)$. It follows analogously when $\deg(Q) = 2p + 1$.

For the family (9)

$$y = 0$$
$$-cx^{s+p+1} = 0$$

Then we can see that $(0, 0)$ is the only critical point. For the family (11)

$$y = 0$$
$$b(m + p + 1)yx^p - b^2mx^{2p+1} = 0$$

Then $b^2mx^{2p+1} = 0$. Again $(0, 0)$ is the only critical point. For the family (12)

$$y = 0$$
$$a(m + s + 1)yx^s - amx^{2s+1} = 0$$

then $amx^{2s+1} = 0$, therefore $(0, 0)$ is the only critical point.

$\square$

Proposition 5.4. *For the family of systems* (6) *the following statements hold:*

a. *If $m + k - 1$ is even and $a(2m + k) > 0$, then $(0, 0)$ is an unstable node.*
b. *If $m + k - 1$ is odd, then $(0, 0)$ is the union of one elliptic sector with one hyperbolic sector.*

c. If $m = 2$ and $k = 1/2$ then the critical points $\left(\sqrt[2k]{\frac{-c}{2a^2 m}}, 0 \right)$ is a unstable node.

Proof.

Proof a. & b:

Over the conditions of (1.1):

$(0, 0)$ is an isolated critical point.

$$X(x, y) = 0$$
$$Y(x, y) = (a(2m + k)x^{m+k-1} + b(2m - k)x^{m-k-1})y - (a^2 m x^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1}$$

The degree of $Y(x, y)$ should be greater than 1.

$$y = F(x) = 0$$

$$f(x) = Y(x, F(x)) - (a^2 m x^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1} = -a^2 m x^{2m+2k-1}\left(1 + \frac{c}{a^2 m}x^{-2k} + \frac{b^2}{a^2}x^{4k}\right)$$

$$\phi(x) = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\Big|_{(x, F(x))} = a(2m + k)x^{m+k-1}\left(1 + \frac{b(2m-k)}{a(2m+k)}x^{-2k}\right)$$

then
$$\alpha = 2m + 2k - 1$$
$$\beta = m + k - 1$$
$$\bar{a} = -a^2 m$$
$$\bar{b} = a(2m + k)$$

Now checking the conditions of the theorem we have:

α is odd, $\bar{a} < 0$, $\bar{b}^2 + 4\bar{a}(\beta + 1) = a^2(2m + k) + 4a^2 m(m + k) > 0$, we have the conditions of item *c)* (1.1).
If β is even and $\bar{b} > 0$, then $(0, 0)$ is an unstable node. On the other hand, if β is odd, then the critical point is the union of an elliptical sector and with an hyperbolic sector.

Proof c.

If $m = 2$ and $k = 1/2$, $c = -2abm$, $c < 0$, $a > 0$, $b > 0$ in this case, taking $d = \sqrt[2k]{\frac{-c}{2a^2 m}}$ and the variable change $u = x - d$ the behavior in a neighborhood of the critical points $\left(\pm \sqrt[2k]{\frac{-c}{2a^2 m}}, 0\right)$, is topologically equivalent to behavior of the system

$$\dot{v} = y$$
$$\dot{y} = [a(2m + k)(v - d)^{m+k-1} + b(2m - k)(v - d)^{m-k-1}]y \qquad (13)$$
$$\quad -a^2 m(v - d)^{2m+2k-1} - c(v - d)^{2m-1} - b^2 m(v - d)^{2m-2k-1},$$

in a neighborhood of the origin. The Jacobian matrix of (13) on the critical points $(0, 0)$ have the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$$

with $\alpha = -8a^2 d^3 - 3cd^2 - 4b^2 d^2$ and $\beta = a\frac{9}{2}d^{3/2} + \frac{7}{2}d^{1/2}$.
Then the eigenvalues are: $\lambda_{1,2} = \beta^2 - 4\alpha = \frac{81}{4}a^2 d^3 + \frac{63}{2}abd^2 + \frac{49}{4}b^2 d + 32a^2 d^3 + 12cd^2 + 16b^2 d^2$.

Now we will define the function

$$H(x) = \frac{209}{4}a^2 x^3 + \frac{63ab + 24c + 32b^2}{2}x^2 + \frac{49}{4}x.$$

Taking into account the condition $c = -2abm$ that is $c = -4ab$, with $a > 0$, $b > 0$ the resultant function is

$$H(x) = \frac{209}{4}a^2 x^3 + \frac{b(32b - 33a)}{2}x^2 + \frac{49}{4}x.$$

Now we going to found the critical points of this function. that is the roots of the equations $H'(x) = 0$. As $H'(x) = \frac{627}{4}a^2x^2 + b(32b - 33a)x + \frac{49}{4}b^2$, then the real roots are

$$x_{1,2} = \frac{2b}{627a^2}\left(-(32b - 33a) \pm b\sqrt{(32b - 33a)^2 - \frac{627 \cdot 49}{4}a^2}\right),$$

whenever $(32b - 33a - \frac{7}{2}\sqrt{627}a)(32b - 33a + \frac{7}{2}\sqrt{627}a) > 0$.

Now $H''(x) = \frac{627}{2}a^2x + b(32b - 33a)$ and $H''(x_1) > 0$ that is x_1 is minimal point of the function H if $(32 - \frac{\sqrt{209}}{2})b$.

Finally according to d values $H(x)$ can take positive or negative values, if $H(d) > 0$ then the origin is a unstable node. If $H(d) < 0$ the origin is a focus for the system (13).

□

6 An example:

Now we show the qualitative study of a particular case associate to the family (7):

$$\dot{x} = y$$

$$\dot{y} = \frac{5}{2}by - \frac{3}{2}b^2x - cx^2.$$

The critical points of this systems are: $(0, 0)$ and $(\frac{-3b^2}{2c}, 0)$. Now we studies the behavior the orbits near them.

For the vector field $X(x, y) = (y, \frac{5}{2}by - \frac{3}{2}b^2x - cx^2)$ the linear part is:

$$DX = \begin{pmatrix} 0 & 1 \\ \frac{-3}{2}b^2 - 2cx & \frac{-5}{2}b \end{pmatrix}.$$

For the origin

$$DX(0, 0) = \begin{pmatrix} 0 & 1 \\ \frac{-3}{2}b^2 & \frac{-5}{2}b \end{pmatrix}$$

and his characteristic polynomials is: $\lambda^2 - \frac{5}{2}b\lambda + \frac{3}{2}b^2 = 0$, then the eigenvalues are $\lambda_1 = \frac{3}{2}b$ and $\lambda_2 = b$. It is the origin is a stable (unstable) node if $b < 0$ ($b > 0$).

Now for the other critical point:

$$DX(\frac{-3b^2}{2c}, 0) = \begin{pmatrix} 0 & 1 \\ \frac{3}{2}b^2 & \frac{5}{2}b \end{pmatrix}$$

and his characteristic polynomials is: $\lambda^2 - \frac{5}{2}b\lambda - \frac{3}{2}b^2 = 0$, then the eigenvalues are $\lambda_1 = 3b$ and $\lambda_2 = \frac{-1}{2}b$. It is, the system has a saddle at the point.

Now we will analysis the infinity behavior, using the Poincaré compactification. For this we will use the equivalent systems at the the chart U_1 and U_2 give by the variable change

$$(x, y) = (\frac{1}{v}, \frac{u}{v}) \qquad (x, y) = (\frac{u}{v}, \frac{1}{v}),$$

respectively. For more see [20].

At U_1 chart the system is:

$$\dot{u} = -u^2 v + \tfrac{5}{2} buv - \tfrac{3}{2} b^2 v - c$$

$$\dot{v} = -uv^2.$$

With critical point $(0, -\tfrac{2c}{3b^2})$, but this is not on the equator of the Poincaré sphere.

At U_2 chart the system is:

$$\dot{u} = v - \tfrac{5}{2} buv + \tfrac{3}{2} b^2 u^2 v + cu^3$$

$$\dot{v} = -\tfrac{5}{2} bv^2 + \tfrac{3}{2} b^2 uv^2 + cu^2 v,$$

with nilpotent singular point at the origin. Using theorem 3.5 ([20, 3.5]), take account that $f(u) = -cu^3 + \ldots + O(u^6)$; $B(u, f) = -c^2 u^5 - \tfrac{5}{2} bc^2 u^6 + O(u^6) + \ldots$; $G(u) = 4cu^2 + \ldots + TOS$; $m = 5$ odd; $a = -c^2$; $n = 2$ even; $m = 2n + 1$; $b^2 + 4a(n + 1) = 4c^2 > 0$. Then $(0, 0)$ is a repelling (attracting) node if $c > 0$ ($c < 0$). Next, we can see the global phase portrait in each case:
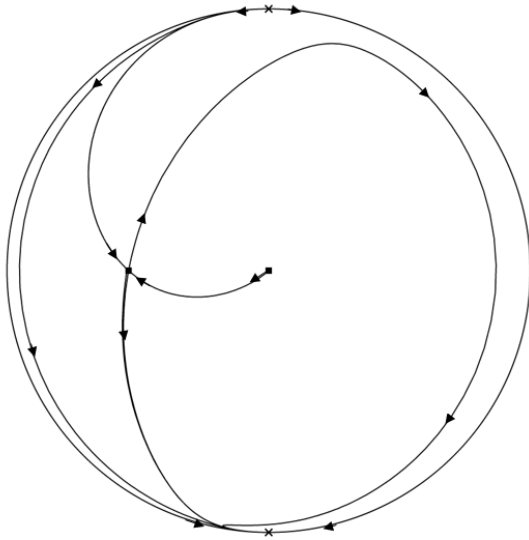

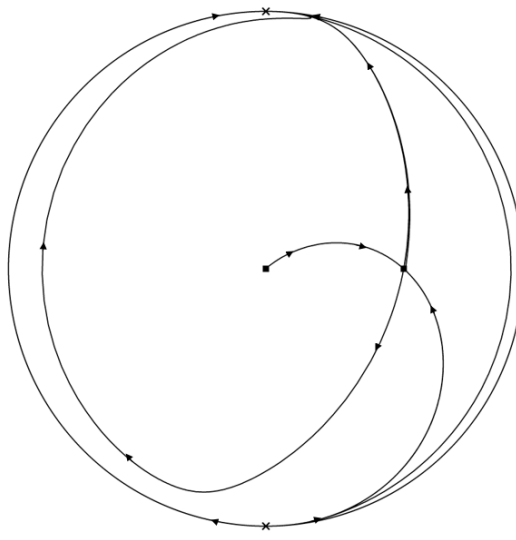**Figure 1:** $b > 0$; $c > 0$.

Figure 2: $b > 0$; $c < 0$.

7 Final Remarks

In this paper we studied from algebraic point of view and singular points study the five parametric family of linear differential systems that came from the corrigendum of Exercise 11 in [18, §1.3.3], which we called Polyanin-Zaitsev vector field. We solved the corrected exercise through a series of transformations using Hamiltonian changes of variables. A analysis was also developed to find critical points and their behavior.
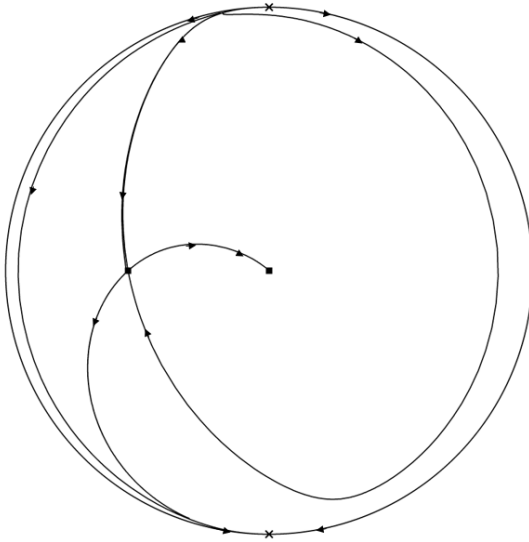
Figure 3: $b < 0$; $c > 0$.
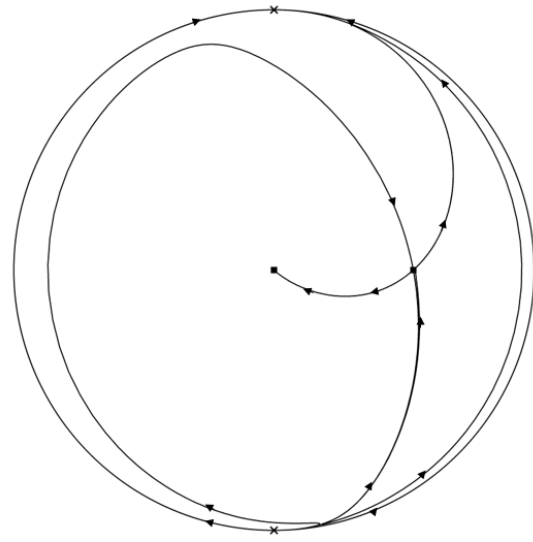
Figure 4: $b < 0$; $c < 0$.

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