THE GENERALIZED FERMAT CONJECTURE

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Abstract. If \(a, b, c\) are non-zero integers, we considerer the fol-
loowing problem: for whic h v alues of \(n\) the line \(ax + by + cz = 0\)
may b e tangen t to the curv e \(x^n + y^n = z^n\)?

We give a partial solution: if \(n = 5\) or if \(n - 1\) is a prime a
umber, then the answ er is the line cannot be tangen t to the curv e.
This problem is strongly related to Fermat’s Last Theorem.

1. Introduction

The classical Fermat Conjecture (was proved to be true [6]) states the
impossibility of finding three integers \(\neq 0\) \(\alpha, \beta, \gamma\) such that \(\alpha^n + \beta^n = \gamma^n\),
where \(n\) is an integers \(\geq 3\). In geometrical terms, the theorem is equivalent
to say that the Fermat curve \(x^n + y^n = z^n\), where \(n \geq 3\), contains no points
whose coordinates in the projective plane over \(\mathbb{C}\) can be expressed in the
form \([\lambda : \mu : \nu]\), where \(\lambda, \mu, \nu\) are non-zero rational numbers. If \(F\) is a field
extension of \(\mathbb{Q}\), we shall say that a point \(P\) in the projective plane over
\(\mathbb{C}\) is an \(F\)–point if there exist elements \(\lambda, \mu, \nu \in F\) not all zero, such that
\(P = [\lambda : \mu : \nu]\). Thus Fermat’s Theorem states that the curve \(x^n + y^n = z^n\)
contains no \(\mathbb{Q}\)–points for \(n \geq 3\). It is well know that the Fermat curves do
not have singular points and hence every point \([x_0 : y_0 : z_0]\) of the curve
yields a unique tangent line \(x_0^{n-1}x + y_0^{n-1}y = z_0^{n-1}z\). We shall say that a
line \(L\) is an \(F\)–tangent to the Fermat curve \(x^n + y^n = z^n\) if the equation
of \(L\) can be expressed in the form \(\lambda x + \mu y = \nu z\), where \(\lambda, \mu, \nu \in F\) not all
zero and \(L\) is the tangent at some point of the curve. It is obvious that the
tangent at an \(F\)–point of the curve is an \(F\)–tangent but the converse is not

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true: the line $x + y = z$ is a $Q$–tangent of the Fermat curve $x^7 + y^7 = z^7$ but the points of tangency are not $Q$–points: in fact the line, $x + y = z$ is tangent to the curve at the points $(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}, 1)$, $(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}, \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, 1)$ and there is no further intersection of the line with the curve. We can state now the generalized Fermat Conjecture.

Fermat’s Last Theorem (FLT), formulated in 1637, states that no three distinct positive integers $\alpha, \beta$ and $\gamma$ can satisfy the equation

$$\alpha^n + \beta^n = \gamma^n$$

if $n$ is an integer greater than 2.

Generalized Fermat Conjecture (GFC). Let $n$ be a natural number $\geq 3$ which is not congruent to 1 (mod 6); then the Fermat curve $x^n + y^n = z^n$ has no $Q$–tangents.

The main relation between GFC and FLT lies in the impossibility that Fermat curve $x^n + y^n = z^n$ has no $Q$–tangents.

In this paper we shall prove the Generalized Fermat Conjecture for $n = 5$ and for every integer $n \geq 3$ such that $n − 1$ is a prime number.

2. Preliminary

The terminology of [2], [3], [4] and [5], is used throughout.

Let $p$ be a prime number $\geq 3$. We know $[Q(\zeta_p^\frac{1}{p}) : Q] = p − 1$; in fact, $x^{p−1} - x^{p−2} + x^{p−3} − \cdots + 1$ is the minimal polynomial of $\zeta_p$ over $Q$. Using this fact, we can prove the following result:

**Proposition 2.1.** Let $p$ be a prime number $\geq 3$. Then $[Q(\cos \frac{\pi}{p}) : Q] = p − 1$.

**Proof.** It is easy to prove that $Q(\zeta_p) = Q(\cos \frac{\pi}{p}, i \sin \frac{\pi}{p})$. So:

$$p − 1 = [Q(\zeta_p) : Q] = [Q(\cos \frac{\pi}{p}, i \sin \frac{\pi}{p}) : Q] = [Q(\cos \frac{\pi}{p}) : Q][Q(\cos \frac{\pi}{p}) : Q].$$

The second degree polynomial $x^2 + 1 - \cos^2 \frac{\pi}{p} \in Q(\cos \frac{\pi}{p})[x]$ has the number $i \sin \frac{\pi}{p}$ as a root. Since $i \sin \frac{\pi}{p} \notin Q(\cos \frac{\pi}{p})$, this polynomial is

1Let us denote by $\zeta_p$ the primitive $p$th root of unity given by $e^{\frac{i\pi}{p}}$. 
irreducible in $\mathbb{Q}\left(\cos\frac{\pi}{p}\right)[x]$ and, therefore it is precisely the minimal polynomial of $i\sin\frac{\pi}{p}$ over $\mathbb{Q}\left(\cos\frac{\pi}{p}\right)$. Therefore:

$$\left[\mathbb{Q}\left(\cos\frac{\pi}{p}, i\sin\frac{\pi}{p}\right) : \mathbb{Q}\left(\cos\frac{\pi}{p}\right)\right] = 2$$

and, by the tower law, we have

$$\left[\mathbb{Q}\left(\cos\frac{\pi}{p}\right) : \mathbb{Q}\right] = \frac{p-1}{2},$$

as wanted. \hfill \Box

**Remark 2.2.** Since $\mathbb{Q}\left(\cos\frac{\pi}{p} + i\sin\frac{\pi}{p}\right) = \mathbb{Q}\left(\cos\frac{n\pi}{p} + i\sin\frac{n\pi}{p}\right)$ for every $n \in \{1, 2, \ldots, p-1\}$, we have $\left[\mathbb{Q}\left(\cos\frac{n\pi}{p}\right) : \mathbb{Q}\right] = \frac{p-1}{2}$ if $n \not\equiv 0 \pmod{p}$.

The Chebyshev polynomials $S_m(x)$ ($m = 0, 1, 2, \ldots$) (see [1]) are defined recursively as follows:

\[
\begin{align*}
S_0(x) &= 0 \\
S_1(x) &= 1 \\
S_m(x) &= xS_{m-1}(x) - S_{m-2}(x) \quad \text{for } m \geq 2.
\end{align*}
\]

**Lemma 2.3.** $\deg(S_m) = m - 1$ ($m = 1, 2, \ldots$) and for every $m \geq 1$ and $\theta \in (0, \pi)$, $S_m(2\cos\theta) = \frac{\sin m\theta}{\sin \theta}$.

**Proof.** The sentence about the degrees is clear from the definition. For the second part, observe that $S_1(2\cos\theta) = \frac{\sin \theta}{\sin \theta} = 1$. Inductively, suppose
Lemma 2.4. Let \( p \) be a prime number \( \geq 3 \) and let \( j, k \) be non-zero integers which are not divisible by \( p \). Then the number \( \frac{\sin kj\pi}{\sin \frac{j\pi}{p}} \) is rational if only if \( k \equiv \pm 1 \pmod{p} \).

Proof. Suppose \( k \not\equiv \pm 1 \pmod{p} \). Let \( k_0 \in \{2, 3, \ldots, p-2\} \) be such that \( k_0 \equiv k \pmod{p} \). Then

\[
\frac{\sin \frac{k_0j\pi}{p}}{\sin \frac{j\pi}{p}} = \pm \frac{\sin \frac{kJ\pi}{p}}{\sin \frac{j\pi}{p}}.
\]

If \( \lambda = \frac{\sin \frac{k_0j\pi}{p}}{\sin \frac{j\pi}{p}} \), then by Lemma 2.3, \( 2 \cos \frac{j\pi}{p} \) is a root of the polynomial \( S_{k_0}(x) - \lambda \). Since \( \sin \frac{k_0j\pi}{p} = \sin \left(\frac{(p-k_0)j\pi}{p}\right) \), the polynomial \( S_{p-k_0}(x) - \lambda \) has also the number \( 2 \cos \frac{j\pi}{p} \) as a root. If \( \lambda \) where rational, (Proposition 2.1) would imply:

\[
k_0 - 1 = \deg(S_{k_0}(x) - \lambda) \geq \frac{p - 1}{2}
\]

and

\[
p - k_0 - 1 = \deg(S_{p-k_0}(x) - \lambda) \geq \frac{p - 1}{2}
\]

Adding these two equations, we would obtain \( p - 2 \geq p - 1 \), a contradiction. Hence the number \( \lambda \) has to be irrational. \( \Box \)
3. Main result

We prove the generalized Fermat conjecture for the special case \( n = 5 \) and for every integer \( n \geq 3 \) such that \( n - 1 \) is a prime number.

**Theorem 3.1** (Generalized Fermat Conjecture). Let \( \lambda, \mu \) be non-zero rational numbers and let \( n \) be a natural number such that \( n - 1 \) is prime or \( n = 5 \). Then the line \( L: \lambda x + \mu y = z \) is not tangent to the Fermat curve of degree \( n \).

**Proof.** Suppose, on the contrary, that \( L \) is tangent to \( C: x^n + y^n = z^n \) and let \([x_0 : y_0 : 1]\) be a point of tangency. We shall prove this point is rational, contradicting Fermat’s theorem. We have then \( x_0^{n-1} = \lambda \) and \( y_0^{n-1} = \mu \). If we set \( w = \cos \frac{\pi}{n-1} + i \sin \frac{\pi}{n-1} \). It is easy to prove that \( w^0 = 1, w^2, \ldots, w^{2n-4} \) is the complete list roots of unity of order \( n - 1 \) and \( w^1, w^3, \ldots, w^{2n-3} \) is the complete list of roots of \(-1\) of order \( n - 1 \). Therefore, there exists two integers \( j, k \in \{0, 1, \ldots, 2n - 3\} \) such that \( x_0 = w^j|\lambda|^{\frac{1}{n-1}} \) and \( y_0 = w^k|\mu|^{\frac{1}{n-1}} \). Observe that if \( x_0 \) and \( y_0 \) are not real numbers, then we should have \( j \neq k \). Since \( \lambda x_0 + \mu y_0 = 1 \), we have then the following equation:

\[
(3.1) \quad w^j(\lambda|\lambda|^{\frac{1}{n-1}}) + w^k(\mu|\mu|^{\frac{1}{n-1}}) = 1.
\]

With no loss of generality, we may suppose that \( j \leq k \). We prove first that the numbers \( x_0, y_0 \) are both real numbers, that is, the only possible values of \( j, k \) are 0 or \( n - 1 \). Indeed, if \( k \neq 0, n - 1 \), then also \( j \neq 0, n - 1 \) and we would have a second equation taking conjugates:

\[
(3.2) \quad w^{-j}(\lambda|\lambda|^{\frac{1}{n-1}}) + w^{-k}(\mu|\mu|^{\frac{1}{n-1}}) = 1.
\]

Adding and subtracting (3.1) and (3.2), we would have

\[
\cos \frac{\pi j}{n-1} \lambda|\lambda|^{\frac{1}{n-1}} + \cos \frac{\pi k}{n-1} \mu|\mu|^{\frac{1}{n-1}} = 1.
\]

\[
\sin \frac{\pi j}{n-1} \lambda|\lambda|^{\frac{1}{n-1}} + \sin \frac{\pi k}{n-1} \mu|\mu|^{\frac{1}{n-1}} = 0.
\]

The determinant of this system is:

\[
\sin \frac{\pi k}{n-1} \cos \frac{\pi j}{n-1} - \sin \frac{\pi j}{n-1} \cos \frac{\pi k}{n-1} = \sin \frac{n}{n-1} \pi (k-j)
\]

and it is equal to zero only if \( k = j \) or \( k = j + (n-1) \). But then \( w^j = \pm w^k \) and equation (3.1) could be written as follows:

\[
w^k(\pm \lambda|\lambda|^{\frac{1}{n-1}} + \mu|\mu|^{\frac{1}{n-1}}) = 1
\]
and \(w^k\) would be a real number, which is a contradiction. Therefore, \(\sin \frac{\pi(k-j)}{n-1} \neq 0\). Applying Cramer’s rule, we obtain:

\[
\lambda|\lambda|^{\frac{i}{r+1}} = \frac{\sin \frac{\pi k}{n-1}}{\sin \frac{\pi (k-j)}{n-1}} \quad \mu|\mu|^{\frac{i}{r+1}} = -\frac{\sin \frac{\pi j}{n-1}}{\sin \frac{\pi (k-j)}{n-1}}.
\]

If \(n - 1\) is a prime number \(p \geq 3\), then:

\[
(3.3) \quad \lambda|\lambda|^{\frac{t}{p}} = \frac{\sin \frac{\pi k}{p}}{\sin \frac{\pi (k-j)}{p}} \quad \mu|\mu|^{\frac{t}{p}} = -\frac{\sin \frac{\pi j}{p}}{\sin \frac{\pi (k-j)}{p}}.
\]

We know \(|Q(w) : Q| = p - 1\). It is obvious that \(w^k + w^{p-k} = 2i \sin \frac{k\pi}{p}\) for each integer \(k\). Therefore, both numbers \(\lambda|\lambda|^{\frac{t}{p}}\) and \(\mu|\mu|^{\frac{t}{p}}\) belong to \(Q(w)\) and, for this reason, the degrees \(|Q(\lambda|\lambda|^{\frac{t}{p}}) : Q|\) and \(|Q(\mu|\mu|^{\frac{t}{p}}) : Q|\) are both \(\leq p - 1\). Since the only possible values of \(|Q(t^\frac{t}{p}) : Q|\), for \(t\) a positive rational, are 1 or \(p\), we conclude that \(|\lambda|^{\frac{t}{p}}\) and \(|\mu|^{\frac{t}{p}}\) are both rational numbers. The trigonometric quotients in (3.3) are then rational numbers.

Since \(p\) is a prime number, there exist integers \(s_1\) and \(s_2\) such that:

\[
s_1(k-j) \equiv k \pmod{p} \\
s_2(k-j) \equiv j \pmod{p}
\]

By Lemma 2.4, we deduce \(s_1, s_2 \equiv \pm 1\ (\text{mod} \ p)\). But then \(\sin \frac{\pi k}{p} = \sin \frac{\pi s_1}{p}\), which, as \(j \neq k\) should imply \(k - j = mp\) for some positive integer \(m\); however as \(k, j \leq 2n - 3\), the only possibility is to have \(k - j = p\) which a contradiction.

If \(n = 5\), then \(w = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}(1 + i)\) and

\[
\lambda|\lambda|^{\frac{t}{4}} = \frac{\sin \frac{\pi k}{4}}{\sin \frac{\pi (k-j)}{4}}, \quad \mu|\mu|^{\frac{t}{4}} = -\frac{\sin \frac{\pi j}{4}}{\sin \frac{\pi (k-j)}{4}}.
\]

The only possible values of \(\sin \frac{\pi k}{4}\) with \(1 \leq k \leq 7\) are \(0, \pm 1, \pm \frac{\sqrt{2}}{2}\). For example if \(k = 6\) and \(j = 1\), give \(\lambda|\lambda|^{\frac{t}{4}} = \frac{\sin \frac{\pi 6}{4}}{\sin \frac{\pi 5}{4}} = \sqrt{2}\) and \(\mu|\mu|^{\frac{t}{4}} = -\frac{\sin \frac{\pi 1}{4}}{\sin \frac{\pi 5}{4}} = 1\). If \(\lambda > 0\) and \(\mu > 0\), necessarily \(j\) is odd, \(k\) is even, \(k - j > 4\), \(j < 4\), \(j \neq 3\) and \(k > 4\). If \(\lambda < 0\) and \(\mu < 0\), then \(j, k\) are odd numbers, \(k - j < 4\), \(j < 4\) and \(k > 4\). The only possibility is \(k = 5\) and \(j = 3\). But then \(\lambda|\lambda|^{\frac{t}{4}} = \frac{\sqrt{2}}{2} = |\mu|^{\frac{t}{4}}\) and \(\lambda = -\frac{\sqrt{1}}{4} = \mu\) contradicting the rationality of \(\lambda\) and \(\mu\). If \(\lambda < 0\) and \(\mu > 0\), then \(j\) is odd, \(k\) is even, \(k - j < 4\), \(k > 4\) and \(j > 4\). The only possibility is \(j = 5\) and \(k = 6\). Then \(\lambda|\lambda|^{\frac{t}{4}} = \sqrt{2}\) and \(\lambda = -\sqrt{4}\), contradicting again the rationality of \(\lambda\). Finally, if \(\lambda > 0\)
and $\mu < 0$, then $j$ is even, $k$ is odd, $k - j < 4, j < 4, k < 4$. The only possibility is $j = 2$ and $|\mu|^2 = \sqrt{2}$ and $\mu = -\sqrt{2}$, contradicting again the rationality of $\mu$.

We have proved then that the only possible values of $j, k$ are $0, n - 1$. Hence, $x_0 = \pm|\lambda|^\frac{1}{n-1}$ and $y_0 = \pm|\mu|^\frac{1}{n-1}$.

The equation (3.1) may therefore be written as

$$\pm|\lambda|^\frac{1}{n-1} + \pm|\mu|^\frac{1}{n-1} = 1.$$ 

Let us say, to fix ideas, that

$$\lambda|\lambda|^\frac{1}{n-1} + \mu|\mu|^\frac{1}{n-1} = 1.$$ 

Setting $\alpha = \mu|\mu|^\frac{1}{n-1}$, we deduce $\alpha$ is a common root of the rational polynomials $\varphi(x) = x^{n-1} - \mu^{n-1}|\mu|$ and $\psi(x) = (1 - x)^{n-1} - \lambda^{n-1}|\lambda|$. If $n - 1$ is an odd prime number, $\alpha$ will be the only common root of $\varphi(x)$ and $\psi(x)$, because if we had another common root, this would be of the form $w^k\mu|\mu|^\frac{1}{n-1}$, with $k = 1, \ldots, n - 2$ and we would have on substituting in $\psi(x)$, an equation of the form:

$$w^j\lambda|\lambda|^\frac{1}{n-1} + w^k\mu|\mu|^\frac{1}{n-1} = 1$$

with $j, k \neq 0, n - 1$, which we have already proved is impossible. If $n - 1 = 2$, the polynomials $\varphi(x)$ and $\psi(x)$ have degree 2 and therefore they could not have another common root. If $n - 1 = 4$, then $-\alpha$ cannot be a root of $\psi(x)$, because in that case $(1 - \alpha)^4 = (1 + \alpha)^4$ and this would imply that $\alpha = 0$. Reasoning as before, we deduce that $w^k\alpha$, with $k \neq 0, 4$, cannot be a root of $\psi(x)$. Therefore, in any situation, $x - \alpha$ must be the greatest common divisor of $\varphi(x)$ and $\psi(x)$. But $\varphi(x)$ and $\psi(x)$ are both rational polynomials and so its greatest common divisor must also be a rational polynomial. We conclude then that $\alpha$ is a rational number, so also $|\mu|^\frac{1}{n-1}$ must be rational. In similar way, we can prove that $|\lambda|^\frac{1}{n-1}$ is rational. But then $[x_0 : y_0 : 1]$ is a rational solution of $x^n + y^n = z^n$, contradicting Fermat’s Theorem.

$$\square$$

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References


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